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Markov Processes and Related Fields

On the Equivalence Between Liggett Duality of Markov Processes and the Duality Relation Between Their Generators

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Abstract. A general result on duality for Feller processes is stated which applies in particular to interacting particle systems. A rather natural condition is presented which ensures that the duality relation between the generator of a conservative Feller process with compact state space and the rate matrix of a Markov chain is equivalent to the according duality relation at the semigroup level. These findings are valid for general duality functions.

KEYWORDS: interacting particle system, duality relation, dual Markov chain

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1. Introduction

The stochastic concept of duality plays an important role in the theory of interacting particle systems (IPS). It allows to relate the evolution of a given interacting particle system to the evolution of another Markov process which is often easier to analyze. The duality technique goes back to Spitzer [14] who used it for the characterization of the stationary distribution of the symmetric exclusion process. Duality for spin-flip processes was employed by Holley and Stroock [6] for the study of the ergodic properties of such processes. Both approaches were systemized by Liggett [9]. There are many recent publications such as [7, 8, 10, 11] where duality is successfully applied.

The idea behind duality shall be illustrated within the following simplified setting. Assume that $(\mathbb{T}_t)_{t\geq 0}$ and $(\mathbb{P}_t)_{t\geq 0}$ are the transition semigroups of two time-continuous Markov processes on $\mathscr{X}$ and $\mathscr{Y}$, respectively, where $\mathscr{X}$ and $\mathscr{Y}$ are finite sets. Let be $H : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$. The two Markov chains are called *H-dual*, if

$$\mathbb{T}_t H(\cdot, B)(\eta) = \mathbb{P}_t H(\eta, \cdot)(B), \quad \eta \in \mathscr{X}, B \in \mathscr{Y}, t \geq 0. \tag{1.1}$$

The infinitesimal characteristics of $(\mathbb{T}_t)_{t\geq 0}$ and $(\mathbb{P}_t)_{t\geq 0}$ shall be denoted by $\mathbb{A}$ and $\mathbb{Q}$, respectively. The rate matrices $\mathbb{A}$ and $\mathbb{Q}$ are *H-dual*, if

$$\mathbb{A} H(\cdot, B)(\eta) = \mathbb{Q} H(\eta, \cdot)(B), \quad \eta \in \mathscr{X}, B \in \mathscr{Y}, \tag{1.2}$$

is fulfilled. One finds easily, that (1.1) implies (1.2). Indeed, one obtains from (1.1) that for fixed $\eta \in \mathscr{X}, B \in \mathscr{Y}$,

$$\frac{1}{t}\big(\mathbb{T}_t H(\cdot, B)(\eta) - H(\eta, B)\big) = \frac{1}{t}\big(\mathbb{P}_t H(\eta, \cdot)(B) - H(\eta, B)\big), \quad t > 0.$$

The limit of the left-hand side as $t \to \infty$ exists and is equal to $\mathbb{A} H(\cdot, B)(\eta)$. The according limit of the right-hand side is equal to $\mathbb{Q} H(\eta, \cdot)(B)$. Thus we get (1.2). To show that the reverse implication is also true, the following idea is pursued. Fix $\eta \in \mathscr{X}$, $B \in \mathscr{Y}$, and define

$$u(t, \eta, B) = \mathbb{T}_t H(\cdot, B)(\eta), \quad t \geq 0,$$

and

$$v(t, B, \eta) = \mathbb{P}_t H(\eta, \cdot)(B), \quad t \geq 0.$$

Then, as $(d/dt)\mathbb{P}_t = \mathbb{Q}\mathbb{P}_t$, it holds that

$$\frac{d}{dt} v(t, B, \eta) = \mathbb{Q} v(t, \cdot, \eta)(B), \quad t > 0; \qquad v(0+, B, \eta) = H(\eta, B)$$

for each $B \in \mathscr{Y}$, $\eta \in \mathscr{X}$. Thus the function $v(\cdot, \cdot, \eta)$ solves for each $\eta \in \mathscr{X}$ the initial value problem

$$\frac{d}{dt} v(t, B) = \mathbb{Q} v(t, \cdot)(B), \quad t > 0; \qquad v(0+, B) = h(B), \tag{1.3}$$

where $h = H(\eta, \cdot)$. As $(d/dt)\mathbb{T}_t = \mathbb{A}\mathbb{T}_t = \mathbb{T}_t\mathbb{A}$, one obtains

$$\frac{d}{dt} u(t, \eta, B) = \mathbb{T}_t \mathbb{A} H(\cdot, B)(\eta), \quad t > 0; \qquad u(0+, \eta, B) = H(\eta, B),$$

for each $\eta \in \mathscr{X}$. Inserting the infinitesimal duality relation (1.2), one finds for the term on the right-hand side that

$$\begin{aligned}
\mathbb{T}_t \mathbb{A} H(\cdot, B)(\eta) &= \mathbb{T}_t\big(\zeta \mapsto \mathbb{A} H(B, \cdot)(\zeta)\big)(\eta) = \mathbb{T}_t\big(\zeta \mapsto \mathbb{Q} H(\zeta, \cdot)(B)\big)(\eta) \\
&= \mathbb{T}_t\Big(\sum_{C \in \mathscr{Y}} \mathbb{Q}(B, C) H(\cdot, C)\Big)(\eta) = \sum_{C \in \mathscr{Y}} \mathbb{Q}(B, C)\mathbb{T}_t H(\cdot, C)(\eta) \\
&= \sum_{C \in \mathscr{Y}} \mathbb{Q}(B, C) u(t, \eta, C) \\
&= \mathbb{Q} u(t, \eta, \cdot)(B), \qquad t > 0, \eta \in \mathscr{X},\ B \in \mathscr{Y}.
\end{aligned}$$

Hence

$$\frac{d}{dt}u(t,\eta,B) = \mathbb{Q}u(t,\eta,\cdot)(B), \quad t>0, \qquad u(0+,\eta,B) = H(\eta,B),$$

for each $\eta \in \mathscr{X}$, $B \in \mathscr{Y}$, that is $u(\cdot,\eta,\cdot)$ is for each $\eta \in \mathscr{X}$ a solution of (1.3). If this initial value problem has for each $\eta \in \mathscr{X}$ a unique solution, then the semigroup duality relation (1.1) holds:

$$\mathbb{T}_t H(\cdot,B)(\eta) = u(t,\eta,B) = v(t,\eta,B) = \mathbb{P}_t H(\eta,\cdot)(B)$$

for $t \geq 0$, $\eta \in \mathscr{X}$, $B \in \mathscr{Y}$.

In the context of IPS one wishes to replace $\mathbb{A}$ by the Markov generator A of a Feller semigroup $(T_t)_{t\geq 0}$ on some compact space $(\mathscr{X}, d)$. The state space $\mathscr{Y}$ of the dual chain is usually chosen to be countably infinite. The question that shall be addressed in this study is under which conditions (1.1) and (1.2) are equivalent in this general situation.

In [9, §III.4 and §VIII.1], the equivalence of (1.1) and (1.2) was considered for specific duality functions and specific interacting particle systems, namely spin-flip systems and symmetric exclusion processes. Essentially the same arguments as in [9] were used, for instance, in [7] for the voter-exclusion process, in [8] for an exclusion process with multiple interactions and in [10] for the symmetric two-particle exclusion-eating process. However, the key issue of proving that the initial value problem (1.3) has a unique bounded solution was not handled in these works. This fact was claimed without proof or, in the case of [7], with reference to [4, Thm.1.3]. The latter reference contains an only vaguely related statement about strongly continuous semigroups. Note that the semigroup on the set of bounded functions generated by a rate matrix $\mathbb{Q}$ is, in general, not strongly continuous. In the special case of spin-flip processes, Holley and Stroock [6] employed the martingale problem to derive duality properties. They considered the initial value problem (1.3), as well, and referred to 'obvious modifications' of Lemma (4.1) and Lemma (4.2) in [15] to conclude its unique solvability. However, the cited propositions are about diffusions on $\mathbb{R}^d$. Therefore it is not clear in which way they apply within the IPS-duality context. Despite this weak point, many authors referred to [6] without giving more detailed arguments there. For instance, Lopez and Sanz [11] considered spin process, where a finite number of states is allowed at each lattice site, and rather general duality functions. Although their setting is in many respects more general than that in [6], they referred to this source without presenting arguments that justify this.

In this paper, the equivalence of (1.1) and (1.2) is derived in full detail for general Feller processes with compact state space and general duality functions. Provided that the generator A of the Feller process and some rate matrix $\mathbb{Q}$ satisfy the infinitesimal duality relation (1.2) with respect to a suitable duality function H, a rather natural condition (Q) on $\mathbb{Q}$ ensures that $\mathbb{Q}$ is non-explosive and the Markov chain generated by $\mathbb{Q}$ is H-dual to the process corresponding to A, that is (1.1) holds. This result applies in particular to general interacting particle systems in the sense of Liggett [9, §I.3]. It is valid without restrictions on the structure of the duality function H. Thus a generalization of the above-mentioned method is derived and the previously lacking details in the literature are worked out within a more abstract setting. As a corollary of our considerations, an additional theorem for Feller processes that are generated by a sum of operators is presented. The findings are applied to a couple of IPS examples. Here the goal is to illustrate that the developed condition for duality is manageable for a variety of different IPS.

The paper is organized as follows. After the main concepts on Markov semigroups, interacting particle systems and Markov chains are introduced in Section 2, the equivalence of (1.1) and (1.2) under condition (Q) is proven in Section 3 for general conservative Feller processes on compact sets and general duality functions (Theorem 3.1). This statement is based on Proposition 3.1 where the unique solvability of the initial value problem (1.3) is shown. Prerequisites are Lemmata 3.1–3.4 where the arguments that were outlined above are proven rigorously in the generalized setting. The additional theorem is presented in Corollary 3.1. In Section 4, the results are specialized to apply to specific interacting particle systems.

2. Setup and notations

2.1. Markov semigroups

Let be given a compact, separable metric space $(\mathscr{X}; d)$. Define the space $B(\mathscr{X})$ of all bounded and Borel-measurable real functions on $\mathscr{X}$ equipped with the supremum norm $\|\cdot\|_\infty$. The subspace of all continuous real functions is denoted by $C(\mathscr{X})$.

In accordance with [9], a strongly continuous non-negative contraction semigroup $(T_t)_{t\geq 0}$ on $C(\mathscr{X})$ which satisfies $T_t\mathbb{1} = \mathbb{1}$[1], $t \geq 0$, is called a *Markov semigroup*. Note that each Markov semigroup on $C(\mathscr{X})$ determines a unique Markov process, which is even a *Feller process* [5, Thm.4.2.7].

From the Hille−Yosida Theorem it follows that each Markov semigroup on $C(\mathscr{X})$ has an infinitesimal generator which is a *Markov generator* in the sense of [9, Def.I.2.7]. Conversely, each Markov generator generates a unique Markov semigroup on $C(\mathscr{X})$ [5, Thm.4.2.2]. Thus one can think of a Feller process on $\mathscr{X}$ as being given by its Markov generator. In the case that $\mathscr{X}$ is compact, it is often suitable to construct the Markov generator from a *Markov pregenerator*. A Markov pregenerator is a linear operator $A : \vartheta(A) \to C(\mathscr{X})$ with dense domain $\vartheta(A) \subset C(\mathscr{X})$ which satisfies the following conditions (G1) and (G2).

[1] $\mathbb{1}(\eta) := 1$, $\eta \in \mathscr{X}$.

(G1) $\mathbb{1} \in \vartheta(A)$, $A\mathbb{1} = 0$;

(G2) If $f \in \vartheta(A)$ and $f(\eta_0) = \max\{f(\eta) : \eta \in \mathscr{X}\} \geq 0$ for some $\eta_0 \in \mathscr{X}$, then $Af(\eta_0) \leq 0$.

Condition (G2) is known as *maximum principle*. Note that a Markov pregenerator is closable in $C(\mathscr{X})$ [9, Prop. I.2.5]. Its closure $\overline{A}$ is a Markov generator if and only if A satisfies the following condition (G3) [5, §1.2].

(G3) There is a $\lambda_0 > 0$ such that $\overline{\mathfrak{R}(\lambda_0 I - A)} = C(\mathscr{X})$.

2.2. Generation of interacting particle systems

The Markov processes that will be considered in the following have state spaces $(\mathscr{X}, d)$ which are special function spaces. See the book of T. Liggett [9] for the detailed construction. Below the main terms are outlined.

Suppose that (W, ρ) is a compact metric space. Let be S a countable set. Frequently S is chosen as $\mathbb{Z}^n$, $n \in \mathbb{N}$. There is a metric d on $\mathscr{X} := W^S$ which generates the product topology on $\mathscr{X}$. Note that $(\mathscr{X}, d)$ is a compact metric space. The elements of $\mathscr{X}$ are referred to as *configurations* on S.

In the context of this paper, a Markov process is named *interacting particle system* (*IPS*), if it is a Feller process on $\mathscr{X}$ that is constructed in the way outlined below. An IPS models the random time changes of configurations.

According to Paragraph 2.1, each IPS is uniquely determined by a Markov generator and can be constructed from a Markov pregenerator. The latter is determined by the specification of *local transition rates*. In detail, let be $\mathscr{T} := \{T \subset S : |T| < \infty, T \neq \emptyset\}$[2] and $\mathscr{T}_\emptyset := \{T \subset S : |T| < \infty\} = \mathscr{T} \cup \{\emptyset\}$. Obviously, $\mathscr{T}$ is finite or countably infinite. For $T \in \mathscr{T}$, define the local configuration space $\mathscr{X}_T := W^T$ and the corresponding projection

$$\pi_T : \mathscr{X} \longrightarrow \mathscr{X}_T : \pi_T \eta := (\eta(x))_{x \in T}.$$

Consider the map

$$\tau_T : \mathscr{X} \times \mathscr{X}_T \longrightarrow \mathscr{X} : \tau_T(\eta, v)(x) := \begin{cases} \eta(x), & \text{if } x \notin T, \\ v(x), & \text{if } x \in T. \end{cases}$$

Given a configuration $\eta \in \mathscr{X}$ and a local configuration $v \in \mathscr{X}_T$, the transformation $\tau_T(\eta, v)$ replaces the local configuration $\pi_T \eta$ in T by v. A family $(c_T(\cdot,\cdot))_{T \in \mathscr{T}}$ of functions $c_T : \mathscr{X} \times \mathfrak{B}(\mathscr{X}_T) \to [0, \infty)$ is a family of *transition rate functions*, if the functions c_T satisfy, for each $T \in \mathscr{T}$, the following conditions (C1)–(C3).

(C1) The function $\eta \mapsto c_T(\eta, \Gamma), \eta \in \mathscr{X}$, is measurable for each $\Gamma \in \mathfrak{B}(\mathscr{X}_T)$.

[2] $|T|$ stands for the cardinality of T

(C2) The map $\Gamma \mapsto c_T(\eta,\Gamma)$, $\Gamma \in \mathfrak{B}(\mathscr{X}_T)$, is a finite measure on $\mathfrak{B}(\mathscr{X}_T)$ for each $\eta \in \mathscr{X}$.

(C3) The function $\eta \mapsto \int_{\mathscr{X}_T} f(v)\, c_T(\eta, dv)$, $\eta \in \mathscr{X}$, is continuous for each $f \in C(\mathscr{X}_T)$.

We define

$$c_T(x) := \sup\left\{|c_T(\eta,du) - c_T(\zeta,dv)| : \pi_{S\setminus\{x\}}\eta = \pi_{S\setminus\{x\}}\zeta\right\}, \quad x \in S,\ T \in \mathscr{T},$$

and

$$c_T := \sup_{\eta\in\mathscr{X}} \sum_{u\in\mathscr{X}_T} c_T(\eta, du), \quad T \in \mathscr{T}.$$

A family $c = (c_T(\cdot,\cdot))_{T\in\mathscr{T}}$ of transition rate functions is *admissible*, if

(C4) $\sup_{x\in S}\sum_{T\ni x} c_T < \infty$,

and

(C5) $\sup_{x\in S}\sum_{T\ni x}\sum_{z\neq x} c_T(z) < \infty$.

Next some function spaces are specified which can serve as domains for the pregenerator of the IPS that shall be constructed. A continuous function $f : \mathscr{X} \to \mathbb{R}$ is a *tame function*, if there exist a set $T \in \mathscr{T}$ and a function $f_T : \mathscr{X}_T \to \mathbb{R}$ such that the representation $f = f_T \circ \pi_T$ holds. In this case, the function f does not depend on the coordinates on $S \setminus T$, that is for any $\eta_1, \eta_2 \in \mathscr{T}$ satisfying $\pi_T\eta_1 = \pi_T\eta_2$ one has $f(\eta_1) = f(\eta_2)$. Let the set of all tame functions be denoted by $T(\mathscr{X})$. Note that $T(\mathscr{X})$ is a dense subset of $C(\mathscr{X})$ (see, for instance, [9, §I.3]). Define further

$$D(\mathscr{X}) := \Big\{ f \in C(\mathscr{X}) : \sum_{x\in S} \sup\left\{|f(\eta) - f(\zeta)| : \eta,\zeta \in \mathscr{X},\ \pi_{S\setminus\{x\}}\eta = \pi_{S\setminus\{x\}}\zeta\right\} < \infty \Big\}.$$

Obviously $T(\mathscr{X}) \subset D(\mathscr{X})$ and therefore $D(\mathscr{X})$ is dense in $C(\mathscr{X})$.

Suppose now that a family $c = (c_T(\cdot,\cdot))_{T\in\mathscr{T}}$ of transition rate functions is given. For $T \in \mathscr{T}$, the operator $A_T : C(\mathscr{X}) \to C(\mathscr{X})$ defined by

$$A_T f(\eta) := \int_{\mathscr{X}_T} \big(f(\tau_T(\eta,\zeta)) - f(\eta)\big) c_T(\eta, d\zeta), \quad f \in C(\mathscr{X}),\ \eta \in \mathscr{X},$$

is a bounded linear operator. Indeed, the property (C3) ensures that the function $A_T f$ is continuous, if f is continuous. The linearity of A_T is obvious and the boundedness of A_T follows from

$$|A_T f(\eta)| \le 2\|f\|_\infty \sup_{\zeta\in\mathscr{X}} c_T(\zeta, \mathscr{X}_T) < \infty, \quad f \in C(\mathscr{X}),\ \eta \in \mathscr{X}.$$

The fact $\sup_{\zeta \in \mathscr{X}} c_T(\zeta, \mathscr{X}_T) < \infty$ is a consequence of the continuity of $c_T(\cdot, \mathscr{X}_T)$ on the compact set $\mathscr{X}$. Moreover, it is easy to see that A_T has the properties (G1) and (G2) of a Markov pregenerator. As A_T is bounded, it is a Markov generator.

Next we define an operator $A : D(\mathscr{X}) \to C(\mathscr{X})$ by

$$\begin{aligned} Af(\eta) &:= \sum_{T \in \mathscr{T}} A_T f(\eta) \qquad (2.1) \\ &= \sum_{T \in \mathscr{T}} \int_{\mathscr{X}_T} c_T(\eta, dv) \left[f(\tau_T(\eta, v)) - f(\eta) \right], \quad \eta \in \mathscr{X},\ f \in D(\mathscr{X}). \end{aligned}$$

Note that $\mathscr{T}$ is a directed set with respect to the set inclusion $\subset$, thus the infinite sums above are to be understood in terms of net convergence. By [9, Prop. I.3.2], A is well-defined if c is admissible. Further, according to [9, Thm. I.3.9], the closure of A is a Markov generator which generates a Markov semigroup $(T_t)_{t \geq 0}$ on $C(\mathscr{X})$. The corresponding Markov process with cádlág trajectories is an interacting particle system.

2.3. Generation of continuous-time conservative Markov chains

The set-up in the following corresponds essentially to [2, Ch. 8] and [13, Ch. 2]. Suppose that $\mathscr{Y}$ is a finite or countably infinite set. One says that for $a : \mathscr{Y} \to \mathbb{R}$ the sum

$$\sum_{B \in \mathscr{Y}} a(B) =: a$$

exists and is equal to a, if for each monotonically increasing sequence $(\mathscr{Y}_n)_{n \in \mathbb{N}}$ with $\mathscr{Y}_n \subset \mathscr{Y}$, $|\mathscr{Y}_n| < \infty$ and $\mathscr{Y} = \cup_{n \in \mathbb{N}} \mathscr{Y}_n$ it holds that

$$\lim_{n \to \infty} \sum_{B \in \mathscr{Y}_n} a(B) = a.$$

Define the space $B(\mathscr{Y})$ of all bounded real functions on $\mathscr{Y}$ equipped with the supremum norm $\|\cdot\|_\infty$.

A function $\mathbb{P} : \mathscr{Y} \times \mathscr{Y} \to [0,1]$ is called *transition matrix* or *stochastic matrix*, if $\sum_{C \in \mathscr{Y}} \mathbb{P}(B, C) = 1$ for each $B \in \mathscr{Y}$. Each stochastic matrix $\mathbb{P}$ can be understood as a contraction on $B(\mathscr{Y})$ by

$$\mathbb{P}h(B) := \sum_{C \in \mathscr{Y}} \mathbb{P}(B, C) h(C), \quad h \in B(\mathscr{Y}),\ B \in \mathscr{Y}.$$

If a function $h : \mathscr{Y} \to \mathbb{R}$ is not bounded, but the sum on the right-hand side above converges for each $B \in \mathscr{Y}$, the symbol $\mathbb{P}h$ is used as well for the resulting function on $\mathscr{Y}$.

A family $(\mathbb{P}_t)_{t\geq 0}$ of stochastic matrices is a *conservative transition semigroup* on $\mathscr{Y}$, if $\mathbb{P}_0 = \mathbb{I}$, where $\mathbb{I}$ is the identity matrix, and $\mathbb{P}_{t+s} = \mathbb{P}_t\mathbb{P}_s$, $t, s \geq 0$. A conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ is *continuous*, if, for each $B \in \mathscr{Y}$, $\lim_{s\downarrow 0}(1 - \mathbb{P}_s(B, B)) = 0$. Note that the continuity of the transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ implies that each function $t \mapsto \mathbb{P}_t h(B), t \geq 0$, is continuous, where $h \in B(\mathscr{Y})$ and $B \in \mathscr{Y}$ are fixed. Indeed, by the semigroup property it is sufficient to show right-continuity at zero. Choose $h \in B(\mathscr{Y})$ and $B \in \mathscr{Y}$. Then

$$\begin{aligned}\big|\mathbb{P}_t h(B) - h(B)\big| &\leq 2\|h\|_\infty \sum_{C\in\mathscr{Y}\setminus\{B\}} \mathbb{P}_t(B, C)\\ &= 2\|h\|_\infty(1 - \mathbb{P}_t(B, B)), \quad B \in \mathscr{Y},\end{aligned}$$

hence the function $t \mapsto \mathbb{P}_t h(B)$, $t \geq 0$, is right-continuous at 0.

Each continuous, conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ is associated to a unique Markov chain on $\mathscr{Y}$.

Suppose that we are given a continuous, conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$. Then it is well-known that the *infinitesimal characteristics*

$$\begin{aligned}\mathbb{Q}(B) &:= \lim_{t\downarrow 0} \frac{1}{t}(1 - \mathbb{P}_t(B, B)) \in [0; \infty], \quad B \in \mathscr{Y},\\ \mathbb{Q}(B, C) &:= \lim_{t\downarrow 0} \frac{1}{t}(\mathbb{P}_t(B, C)) \in [0; \infty), \quad B, C \in \mathscr{Y},\end{aligned}$$

exist [2, Thm. 2.1]. For convenience let

$$\mathbb{Q}(B, B) := -\mathbb{Q}(B), \quad B \in \mathscr{Y}.$$

Since the transition semigroup is conservative, one has

$$\sum_{C\in\mathscr{Y}\setminus\{B\}} \mathbb{Q}(B, C) = -\mathbb{Q}(B, B) < \infty, \quad B \in \mathscr{Y}.$$

Hence the infinitesimal characteristics $\mathbb{Q}$ of a continuous, conservative transition semigroup satisfies the conditions (R1)–(R3) below.

A continuous, conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ on $\mathscr{Y}$ with infinitesimal characteristics $\mathbb{Q}$ is said to *satisfy condition* (PQ) if

$$\sum_{C\in\mathscr{Y}} \mathbb{P}_t(B, C)\mathbb{Q}(C) < \infty, \quad B \in \mathscr{Y},\ t > 0. \tag{PQ}$$

A matrix $\mathbb{Q} := (\mathbb{Q}(B, C))_{B,C\in\mathscr{Y}}$ with

$$0 \leq \mathbb{Q}(B, C) < \infty, \quad B, C \in \mathscr{Y},\ B \neq C, \tag{R1}$$

and

$$-\infty < Q(B, B) =: -\mathbb{Q}(B) \leq 0, \quad B \in \mathscr{Y}, \tag{R2}$$

as well as

$$\sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C) = 0, \quad B \in \mathscr{Y}, \tag{R3}$$

is a *rate matrix* or *Q-matrix*. Each rate matrix $\mathbb{Q} := (\mathbb{Q}(B,C))_{B,C\in\mathscr{Y}}$ can be understood as a linear operator on $B(\mathscr{Y})$ by

$$\begin{aligned}\mathbb{Q}h(B) &:= \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)h(C) \\ &= \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[h(C)-h(B)], \quad h \in B(\mathscr{Y}),\ B \in \mathscr{Y}.\end{aligned}$$

This operator is bounded, if the condition

$$\sup_{B\in\mathscr{Y}} \mathbb{Q}(B) < \infty \tag{Q*}$$

holds. If a function $f : \mathscr{Y} \to \mathbb{R}$ is nonnegative then

$$\mathbb{Q}f(B) := \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)f(C) = -\mathbb{Q}(B)f(B) + \sum_{C\in\mathscr{Y}\setminus\{B\}} \mathbb{Q}(B,C)f(C), \quad B \in \mathscr{Y},$$

is well-defined in $[0,\infty]$.

Below it is typically assumed that a rate matrix satisfies the following condition which is weaker than (Q*).

(Q) There are a sequence $(\mathscr{Y}_n)_{n\in\mathbb{N}}$ of subsets $\mathscr{Y}_n \subset \mathscr{Y}$ that increase to $\mathscr{Y}$ as $n \to \infty$, a nonnegative function φ on $\mathscr{Y}$ with $\lim_{n\to\infty} \inf_{B\notin\mathscr{Y}_n} \varphi(B) = \infty$ and a real c such that the following conditions are satisfied.

$$\sup_{B\in\mathscr{Y}_n} \mathbb{Q}(B) < \infty, \quad n \in \mathbb{N}; \tag{Q0}$$

$$\mathbb{Q}\varphi(B) \le c\varphi(B), \quad B \in \mathscr{Y}; \tag{Q1}$$

$$\mathbb{Q}(B) \le c\varphi(B), \quad B \in \mathscr{Y}. \tag{Q2}$$

Condition (Q1) is sufficient for the existence of a unique conservative and continuous transition semigroup $(\mathbb{P}_t)_{t\ge 0}$ with given infinitesimal characteristics $\mathbb{Q}$ [1, Cor. 2.2.16]. According to [1, Prop. 2.2.13], (Q1) is equivalent to the statement

$$\mathbb{P}_t\varphi \le e^{ct}\varphi(B), \quad t \ge 0,\ B \in \mathscr{Y}. \tag{P1}$$

If a rate matrix $\mathbb{Q}$ satisfies condition (Q), then it fulfills condition (PQ). Indeed, from (Q1) and (Q2) via (P1) one has

$$\sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)\mathbb{Q}(C) \le c \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)\varphi(C) \le ce^{ct}\varphi(B) < \infty,$$

where $t \ge 0, B \in \mathscr{Y}$. Note that the condition (Q) and thus condition (PQ) are implied by the more restrictive condition (Q^*).

3. A duality criterion

Assume that $(\mathscr{X}; d)$ is a compact metric space and let be $(T_t)_{t\geq 0}$ a Markov semigroup on $C(\mathscr{X})$ with associated Markov generator $(A, \vartheta(A))$. Suppose that $\mathscr{Y}$ is an at most countably infinite set and $(\mathbb{P}_t)_{t\geq 0}$ is a continuous, conservative transition semigroup on $\mathscr{Y}$ with infinitesimal characteristics $\mathbb{Q}$. Further assume that $H : \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ is a bounded function which satisfies

$$H(\cdot, B) \in \vartheta(A), \quad B \in \mathscr{Y}. \tag{3.1}$$

The semigroups $(T_t)_{t\geq 0}$ and $(\mathbb{P}_t)_{t\geq 0}$ and, accordingly, the corresponding Markov processes are said to be *in duality with respect to* H (H*-dual*), if the following condition (D-S) is satisfied.

$$T_t H(\cdot, B)(\eta) = \mathbb{P}_t H(\eta, \cdot)(B), \quad \eta \in \mathscr{X},\ B \in \mathscr{Y},\ t \geq 0. \tag{D-S}$$

The generator A of a Markov semigroup $(T_t)_{t\geq 0}$ and the infinitesimal characteristics $\mathbb{Q}$ of a continuous, conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ on $\mathscr{Y}$ *satisfy condition* (D-I), if

$$AH(\cdot, B)(\eta) = \mathbb{Q}H(\eta, \cdot)(B), \quad \eta \in \mathscr{X},\ B \in \mathscr{Y}, \tag{D-I}$$

is fulfilled.

Lemma 3.1. *Suppose that* $f \in B([0,\infty) \times \mathscr{Y})^3$. *If the limit* $\lim_{t\downarrow 0} f(t, B) =: f(0+, B)$ *exists for each* $B \in Y$*, then*

$$\lim_{t\downarrow 0} \frac{\mathbb{P}_t f(t,B) - f(t,B)}{t} = \mathbb{Q}f(0+, B), \quad B \in \mathscr{Y}.$$

Proof. Let $B \in \mathscr{Y}$, $f \in B([0,\infty) \times \mathscr{Y})$ and define $g(t, C) := f(t, C) - f(t, B)$, $t \geq 0$, $C \in \mathscr{Y}$. It is to show that

$$\lim_{t\downarrow 0} \frac{1}{t}\Big(\sum_{C\in\mathscr{Y}} g(t, C)\mathbb{P}_t(B, C)\Big) = \sum_{C\in\mathscr{Y}} g(0+, C)\mathbb{Q}(B, C).$$

Let some sequence $(\mathscr{Y}_n)_{n\in\mathbb{N}}$ be given, the $\mathscr{Y}_n$'s being finite subsets of $\mathscr{Y}$ which increase towards $\mathscr{Y}$ as $n \to \infty$. There exists $n_0 \in \mathbb{N}$ such that $B \in \mathscr{Y}_n$ for all $n > n_0$. Thus one obtains for $n > n_0$, $t > 0$,

$$\begin{aligned} \mathbb{P}_t g(t, B) &= \sum_{C\in\mathscr{Y}} g(t, C)\mathbb{P}_t(B, C) \\ &\geq \sum_{C\in\mathscr{Y}_n} g(t, C)\mathbb{P}_t(B, C) - \|g\|_\infty \sum_{C\in\mathscr{Y}\setminus\mathscr{Y}_n} \mathbb{P}_t(B, C) \\ &= \sum_{C\in\mathscr{Y}_n} g(t, C)\mathbb{P}_t(B, C) - \|g\|_\infty \Big((1 - \mathbb{P}_t(B, B)) - \sum_{C\in\mathscr{Y}_n\setminus\{B\}} \mathbb{P}_t(B, C)\Big). \end{aligned}$$

[3] For general measurable spaces E, the symbol $B(E)$ denotes the space of real-valued, bounded and measurable functions on E.

Both sums at the right-hand side above are finite, thus one concludes

$$\lim_{t\downarrow 0}\sum_{C\in\mathscr{Y}_n} g(t,C)\frac{1}{t}\mathbb{P}_t(B,C)=\sum_{C\in\mathscr{Y}_n} g(0+,C)\mathbb{Q}(B,C)$$

and

$$\lim_{t\downarrow 0}\sum_{C\in\mathscr{Y}_n\setminus\{B\}}\frac{1}{t}\mathbb{P}_t(B,C)=\sum_{C\in\mathscr{Y}_n\setminus\{B\}}\mathbb{Q}(B,C).$$

Consequently,

$$\begin{aligned}\liminf_{t\downarrow 0}&\frac{1}{t}\mathbb{P}_t g(t,B)\\ &\geq\sum_{C\in\mathscr{Y}_n} g(0+,C)\mathbb{Q}(B,C)-\|g\|_\infty\Big(\mathbb{Q}(B,B)-\sum_{C\in\mathscr{Y}_n\setminus\{B\}}\mathbb{Q}(B,C)\Big)\end{aligned}$$

for $n>n_0$. The term $\mathbb{Q}(B,B)-\sum_{C\in\mathscr{Y}_n\setminus\{B\}}\mathbb{Q}(B,C)$ tends to 0 as $n\to\infty$, therefore

$$\liminf_{t\downarrow 0}\frac{1}{t}\mathbb{P}_t g(t,B)\geq\sum_{C\in\mathscr{Y}} g(0+,C)\mathbb{Q}(B,C).$$

Analogously, the estimate

$$\begin{aligned}\mathbb{P}_t g(t,B)&=\sum_{C\in\mathscr{Y}} g(t,C)\mathbb{P}_t(B,C)\\ &\leq\sum_{C\in\mathscr{Y}_n} g(t,C)\mathbb{P}_t(B,C)+\|g\|_\infty\sum_{C\in\mathscr{Y}\setminus\mathscr{Y}_n}\mathbb{P}_t(B,C)\end{aligned}$$

yields

$$\limsup_{t\downarrow 0}\frac{1}{t}\mathbb{P}_t g(t,B)\leq\sum_{C\in\mathscr{Y}} g(0+,C)\mathbb{Q}(B,C).$$

□

Lemma 3.2. *If $f\in B([0,\infty)\times\mathscr{Y})$ is right-continuous with respect to the first argument, then*

$$\lim_{s\to t}\mathbb{P}_s f(s,B)=\mathbb{P}_t f(t,B),\quad t>0,\ B\in\mathscr{Y}.$$

Proof. Fix $B\in\mathscr{Y}$, $f\in B([0,\infty)\times\mathscr{Y})$ and choose a sequence $(\mathscr{Y}_n)_{n\in\mathbb{N}}$, the $\mathscr{Y}_n$'s being finite subsets of $\mathscr{Y}$ which increase towards $\mathscr{Y}$ as $n\to\infty$. Then, for sufficiently large n such that $B\in\mathscr{Y}_n$,

$$\mathbb{P}_s f(s,B)\geq\sum_{C\in\mathscr{Y}_n} f(s,C)\mathbb{P}_s(B,C)-\|f\|_\infty\Big(1-\sum_{C\in\mathscr{Y}_n}\mathbb{P}_s(B,C)\Big),$$

which yields

$$\liminf_{s\to t} \mathbb{P}_s f(s,B) \geq \sum_{C\in\mathscr{Y}_n} f(t,C)\mathbb{P}_t(B,C) - \|f\|_\infty\Big(1 - \sum_{C\in\mathscr{Y}_n} \mathbb{P}_t(B,C)\Big).$$

Taking the limit $n\to\infty$ on the right-hand side, one obtains

$$\liminf_{s\to t} \mathbb{P}_s f(s,B) \geq \sum_{C\in\mathscr{Y}} f(t,C)\mathbb{P}_t(B,C).$$

Analogously,

$$\limsup_{s\to t} \mathbb{P}_s f(s,B) \leq \sum_{C\in\mathscr{Y}} f(t,C)\mathbb{P}_t(B,C).$$

□

Lemma 3.3. *Let $B\in\mathscr{Y}$ and $h\in B(\mathscr{Y})$. Then the function $t\mapsto \mathbb{P}_t h(B)$, $t>0$, is continuously differentiable and the derivative satisfies*

(i)

$$\frac{d}{dt}\mathbb{P}_t h(B) = \mathbb{Q}(\mathbb{P}_t h)(B) = \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[\mathbb{P}_t h(C) - \mathbb{P}_t h(B)], \quad t>0.$$

(ii) *If the transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ satisfies the condition* (PQ), *then*

$$\begin{aligned}\frac{d}{dt}\mathbb{P}_t h(B) &= \mathbb{P}_t(\mathbb{Q}h)(B)\\ &= \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)\sum_{V\in\mathscr{Y}} \mathbb{Q}(C,V)[h(V)-h(C)], \quad t>0.\end{aligned}$$

Remark 3.1. A proof of the above lemma can be found in [3, §13.5], where it was used to derive an integration by parts formula for jump processes. Nevertheless an independent proof is given here to make the argumentation self-contained.

Proof.
(i) Let $t>0$, $B\in\mathscr{Y}$ and $h\in B(\mathscr{Y})$. The semigroup property of $(\mathbb{P}_s)_{s\geq 0}$ gives

$$\frac{1}{s}\big(\mathbb{P}_{t+s}h(B) - \mathbb{P}_t h(B)\big) = \frac{1}{s}\sum_{C\in\mathscr{Y}} \mathbb{P}_s(B,C)[\mathbb{P}_t h(C) - \mathbb{P}_t h(B)], \quad s>0.$$

Define

$$u_{h,B}(t,\cdot) := \mathbb{P}_t h(\cdot) - \mathbb{P}_t h(B).$$

Clearly, $u_{h,B}(t,\cdot) \in B(\mathscr{Y})$ and $u_{h,B}(t,B)=0$. Applying Lemma 3.1 to $u_{h,B}(t,\cdot)$, one obtains

$$\begin{aligned}
&\lim_{s\downarrow 0} \frac{1}{s} \sum_{C\in\mathscr{Y}} \mathbb{P}_s(B,C)[\mathbb{P}_t h(C) - \mathbb{P}_t h(B)] \\
&\quad = \lim_{s\downarrow 0} \frac{1}{s} \mathbb{P}_s u_{h,B}(t,\cdot)(B) = \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C) u_{h,B}(t,C) \\
&\quad = \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[\mathbb{P}_t h(C) - \mathbb{P}_t h(B)].
\end{aligned}$$

Hence

$$\lim_{s\downarrow 0} \frac{1}{s}(\mathbb{P}_{t+s} h(B) - \mathbb{P}_t h(B)) = \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[\mathbb{P}_t h(C) - \mathbb{P}_t h(B)].$$

This shows that the continuous function $t \mapsto \mathbb{P}_t h(B)$, $t > 0$, has at each $t > 0$ the right-hand side derivative

$$\sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C) u_{h,B}(t,C).$$

As $u_{h,B}(\cdot,C)$ is continuous and uniformly bounded by $2\|h\|_\infty$ and because of

$$\sum_{C\in\mathscr{Y}\setminus\{B\}} \mathbb{Q}(B,C) = \mathbb{Q}(B) < \infty,$$

this right-hand side derivative of $t \mapsto \mathbb{P}_t h(B)$, $t > 0$ is continuous. Each continuous function with continuous right-hand side derivative is differentiable and its derivative agrees with its right-hand side derivative. This proves (i).

(ii) Suppose that $(\mathbb{P}_t)_{t\geq 0}$ satisfies (PQ). For $B \in \mathscr{Y}$ and $h \in B(\mathscr{Y})$, one obtains from the semigroup property of $(\mathbb{P}_t)_{t\geq 0}$ that

$$\begin{aligned}
&(\mathbb{P}_{t+s}\, h(B) - \mathbb{P}_t h(B)) \\
&\quad = \mathbb{P}_t(\mathbb{P}_s h(\cdot) - h(\cdot))(B) = \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)(\mathbb{P}_s h(C) - h(C)) \\
&\quad = \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C) \sum_{V\in\mathscr{Y}} \mathbb{P}_s(C,V)[h(V) - h(C)], \quad s,t > 0.
\end{aligned}$$

Obviously

$$\begin{aligned}
\Big| \sum_{V\in\mathscr{Y}} \mathbb{P}_s(C,V)[h(V) - h(C)] \Big| &\leq 2\|h\|_\infty \sum_{V\in\mathscr{Y}\setminus\{C\}} \mathbb{P}_s(C,V) \\
&= 2\|h\|_\infty (1 - \mathbb{P}_s(C,C)), \quad s > 0,\ C \in \mathscr{Y}.
\end{aligned}$$

Each continuous conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ with infinitesimal characteristics $\mathbb{Q}$ satisfies

$$\frac{1}{s}(1-\mathbb{P}_s(C,C)) \leq \mathbb{Q}(C), \quad s>0,\ C\in\mathscr{Y},$$

see, for instance [2, 8.(2.15)]. Hence

$$\frac{1}{s}\Big|\sum_{V\in\mathscr{Y}}\mathbb{P}_s(C,V)[h(V)-h(C)]\Big| \leq 2\|h\|_\infty\mathbb{Q}(C), \quad s>0,\ C\in\mathscr{Y}.$$

By (PQ), it follows that

$$\begin{aligned}\frac{1}{s}\sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\Big|\sum_{V\in\mathscr{Y}}\mathbb{P}_s(C,V)[h(V)-h(C)]\Big|\\ \leq 2\|h\|_\infty\sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\mathbb{Q}(C)<\infty\end{aligned}$$

for $s,t>0$. According to Lemma 3.1, it holds for each $C\in\mathscr{Y}$ that

$$\lim_{s\downarrow 0}\Big(\frac{1}{s}\sum_{V\in\mathscr{Y}}\mathbb{P}_s(C,V)[h(V)-h(C)]\Big) = \sum_{V\in\mathscr{Y}}\mathbb{Q}(C,V)[h(V)-h(C)].$$

Hence

$$\begin{aligned}&\lim_{s\downarrow 0}\frac{1}{s}\sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\sum_{V\in\mathscr{Y}\setminus\{C\}}\mathbb{P}_s(C,V)[h(V)-h(C)]\\ &\quad=\sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\lim_{s\downarrow 0}\frac{1}{s}\sum_{V\in\mathscr{Y}\setminus\{C\}}\mathbb{P}_s(C,V)[h(V)-h(C)]\\ &\quad=\sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\sum_{V\in\mathscr{Y}}\mathbb{Q}(C,V)[h(V)-h(C)], \quad t>0.\end{aligned}$$

Thus the right-hand side derivative of the function $t\mapsto\mathbb{P}_t h(B)$, $t>0$, admits for each $t>0$ the representation

$$\lim_{s\downarrow 0}\frac{1}{s}(\mathbb{P}_{t+s}h(B)-\mathbb{P}_t h(B)) = \sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\sum_{V\in\mathscr{Y}}\mathbb{Q}(C,V)[h(V)-h(C)].$$

It is argued in the proof of (i) that the derivative of the function $t\mapsto\mathbb{P}_t h(B)$, $t>0$, exists and agrees with its right-hand side derivative. Hence

$$\frac{d}{dt}\mathbb{P}_t h(B) = \sum_{C\in\mathscr{Y}}\mathbb{P}_t(B,C)\sum_{V\in\mathscr{Y}}\mathbb{Q}(C,V)[h(V)-h(C)], \quad t>0.$$

$\square$

If a function $u \in B((0,\infty) \times \mathscr{Y})$ is differentiable with respect to the first argument, then this partial derivative shall be denoted by

$$u'(t,B) := \frac{d}{dt} u(t,B), \quad B \in \mathscr{Y}.$$

Lemma 3.4. *Let $\mathbb{Q}$ be a rate matrix with associated transition semigroup $(\mathbb{P}_t)_{t\geq 0}$. Suppose that* (PQ) *holds. If a function $u \in B((0,\infty) \times \mathscr{Y})$ is continuously differentiable with respect to the first argument and satisfies*

$$|u'(t,B)| \leq K\mathbb{Q}(B), \quad t > 0,\ B \in \mathscr{Y}, \tag{3.2}$$

for some constant $K > 0$, then for each $B \in \mathscr{Y}$ the function $t \mapsto \mathbb{P}_t u(t,B)$, $t > 0$, is differentiable and its derivative satisfies

$$\frac{d}{dt}\big(\mathbb{P}_t u(t,\cdot)(B)\big) = \frac{d}{ds}\big(\mathbb{P}_s u(t,\cdot)(B)\big)\Big|_{s=t} + \mathbb{P}_t u'(t,\cdot)(B), \quad t > 0.$$

Proof. Fix $t > 0$ and consider a function $u \in B([0,\infty) \times \mathscr{Y})$ that satisfies (3.2) for some constant $K > 0$. Choose $\delta > 0$, $v \in \mathbb{R}$ with $|v| < \delta < t$. One has

$$\begin{aligned} &\mathbb{P}_{t+v}\, u(t+v,\cdot) - \mathbb{P}_t u(t,\cdot) \\ &\qquad = \mathbb{P}_t(u(t+v,\cdot) - u(t,\cdot)) + (\mathbb{P}_{t+v} u(t+v,\cdot) - \mathbb{P}_t u(t+v,\cdot)). \end{aligned}$$

Fix $B \in \mathscr{Y}$ and consider

$$\frac{1}{v}\mathbb{P}_t(u(t+v,\cdot) - u(t,\cdot)) = \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)\frac{u(t+v,C) - u(t,C)}{v}. \tag{3.3}$$

One finds that

$$\Big|\frac{u(t+v,C) - u(t,C)}{v}\Big| \leq \frac{1}{v}\int_t^{t+v} |u'(s,C)|\, ds \leq K\mathbb{Q}(C), \quad C \in \mathscr{Y},$$

where $\sum_{C\in\mathscr{Y}} \mathbb{P}_v(B,C)\mathbb{Q}(C) < \infty$, $B \in \mathscr{Y}$, according to (PQ). Thus the Dominated Convergence Theorem applies for the limit $v \downarrow 0$ in (3.3). One concludes

$$\lim_{v\downarrow 0} \frac{1}{v} \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)\frac{u(t+v,D) - u(t,D)}{v} = \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B,C)u'(t,C).$$

One verifies easily, that the function $[0,\infty) \times \mathscr{Y} \ni (v,C) \mapsto \mathbb{P}_t u(t+v,\cdot)(C)$ is bounded and satisfies $\lim_{v\to 0} \mathbb{P}_t u(t+v,\cdot)(C) = \mathbb{P}_t u(t,\cdot)(C)$, $C \in \mathscr{Y}$, by the Dominated Convergence Theorem. Therefore, the limit

$$\begin{aligned} &\lim_{v\downarrow 0} \frac{1}{v}\big(\mathbb{P}_v\mathbb{P}_t u(t+v,\cdot) - \mathbb{P}_t u(t+v,\cdot)\big)(B) \\ &\qquad = \mathbb{Q}(\mathbb{P}_t u(t,\cdot))(B) = \frac{d}{ds}\big(\mathbb{P}_s u(t,\cdot)(B)\big)\Big|_{s=t} \end{aligned}$$

exists for each $B \in \mathscr{Y}$ by Lemma 3.1 and Lemma 3.3 (i). For $v < 0$, observe that

$$\mathbb{P}_{t+v}u(t+v,\cdot) - \mathbb{P}_t u(t+v,\cdot) = -\big(\mathbb{P}_{-v}\mathbb{P}_{t+v}u(t+v,\cdot) - \mathbb{P}_{t+v}u(t+v,\cdot)\big).$$

Since $\lim_{v\to 0} \mathbb{P}_{t+v}u(t+v,\cdot)(B) = \mathbb{P}_t u(t,\cdot)(B)$, $B \in \mathscr{Y}$, by Lemma 3.2, one can conclude from Lemma 3.1 applied to the bounded function $[0,\infty) \times \mathscr{Y} \ni (v, C) \mapsto \mathbb{P}_{t+v}u(t+v,\cdot)(C)$ that

$$\lim_{v\uparrow 0} \frac{1}{v}\big(\mathbb{P}_{-v}\mathbb{P}_{t+v}u(t+v,\cdot) - \mathbb{P}_{t+v}u(t+v,\cdot)\big)(B) = \mathbb{Q}(\mathbb{P}_t u(t,\cdot))(B), \quad B \in \mathscr{Y}.$$

Consequently,

$$\lim_{v\to 0} \frac{1}{v}\big(\mathbb{P}_{t+v}u(t+v,\cdot) - \mathbb{P}_t u(t+v,\cdot)\big)(B) = \frac{d}{ds}\big(\mathbb{P}_s u(t,\cdot)(B)\big)\Big|_{s=t}, \quad B \in \mathscr{Y}.$$

□

Proposition 3.1. *Let $\mathbb{Q}$ be a rate matrix with associated transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ and let the condition* (PQ) *be satisfied. Let $h \in B(\mathscr{Y})$ be fixed. Then the initial value problem*

$$\frac{d}{dt}u(t,B) = \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[u(t,C) - u(t,B)], \qquad \text{(IVP)}$$
$$u(0+;B) = h(B), \quad B \in \mathscr{Y},\ t > 0,$$

has a unique solution in $B([0,\infty)\times\mathscr{Y})$ which is given by

$$u(t,B) = \mathbb{P}_t h(B), \quad t \geq 0,\ B \in \mathscr{Y}.$$

Proof. Fix $h \in B(\mathscr{Y})$. The function $u(t,B) := \mathbb{P}_t h(B)$, $t \geq 0$, $B \in \mathscr{Y}$, satisfies (IVP) by Lemma 3.3 (i). Since the semigroup $(\mathbb{P}_t)_{t\geq 0}$ is contractive, $u \in B([0,\infty)\times\mathscr{Y})$.

Suppose that $v \in B([0,\infty)\times\mathscr{Y})$ solves (IVP). Then the function $v(\cdot, B)$ is differentiable for each $B \in Y$. Its derivative is continuous, since it admits a representation as the uniform limit of continuous functions. Indeed, choosing a sequence $(\mathscr{Y}_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathscr{Y}$ which increase towards $\mathscr{Y}$ as $n \to \infty$, one finds that

$$u'(\cdot,B) = \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[u(\cdot,C)-u(\cdot,B)] = \lim_{n\to\infty} \sum_{C\in\mathscr{Y}_n} \mathbb{Q}(B,C)[u(\cdot,C)-u(\cdot,B)].$$

In addition, it holds that

$$|v'(t,B)| \leq \sum_{C\in\mathscr{Y}\setminus\{B\}} \mathbb{Q}(B,C)|v(t,C) - v(t,B)| \leq 2\|v\|_\infty \mathbb{Q}(B), \quad t > 0,\ B \in \mathscr{Y}.$$

Hence Lemma 3.4 applies. One finds that for fixed $B \in \mathscr{Y}$, $t > 0$, the function $s \to \mathbb{P}_{t-s} v(s, B)$, $0 < s < t$, is differentiable and its derivative satisfies

$$\frac{d}{ds}\big(\mathbb{P}_{t-s} v(s,\cdot)(B)\big) = -\frac{d}{dr}\big(\mathbb{P}_r v(s,\cdot)(B)\big)\Big|_{r=t-s} + \mathbb{P}_{t-s} v'(s,\cdot)(B).$$

By (IVP) and Lemma 3.3 (ii),

$$\begin{aligned}\mathbb{P}_{t-s}\Big(\frac{d}{ds}v(s,\cdot)\Big)(B) &= \mathbb{P}_{t-s}(\mathbb{Q}v(s,\cdot))(B)\\ &= \sum_{C\in\mathscr{Y}}\sum_{V\in\mathscr{Y}} \mathbb{P}_{t-s}(B,C)\mathbb{Q}(C,V)[v(s,V)-v(s,C)]\\ &= \frac{d}{dr}\big(\mathbb{P}_r v(s,\cdot)(B)\big)\Big|_{r=t-s}.\end{aligned}$$

Thus

$$\frac{d}{ds}\big(\mathbb{P}_{t-s} v(s,\cdot)(B)\big) = 0,$$

which implies that the map $s \mapsto \mathbb{P}_{t-s} v(s,\cdot)(B)$ equals a constant on $(0, t)$. It follows that

$$\mathbb{P}_t v(0+,\cdot)(B) = \lim_{s\downarrow 0} \mathbb{P}_{t-s} v(s,\cdot)(B) = \lim_{s\uparrow t} \mathbb{P}_{t-s} v(s,\cdot)(B) = v(t,B),$$

where the first and last equality are a consequence of Lemma 3.2. Because of $v(0+,\cdot) = h$ one obtains $v(t,B) = \mathbb{P}_t h(B)$. □

Theorem 3.1. *Let $(T_t)_{t\geq 0}$ be a Markov semigroup on $C(\mathscr{X})$ with infinitesimal generator $(A, \vartheta(A))$ and let H be a real-valued bounded function on $\mathscr{X} \times \mathscr{Y}$ satisfying the condition*

$$H(\cdot, B) \in \vartheta(A), \quad B \in \mathscr{Y}.$$

(a) *Suppose that $(\mathbb{P}_t)_{t\geq 0}$ is a continuous conservative transition semigroup on $\mathscr{Y}$ with infinitesimal characteristics $\mathbb{Q}$. If $(T_t)_{t\geq 0}$ and $(\mathbb{P}_t)_{t\geq 0}$ are in duality with respect to H, then*

$$AH(\cdot,B)(\eta) = \mathbb{Q}H(\eta,\cdot)(B), \quad \eta \in \mathscr{X},\ B \in \mathscr{Y}, \tag{D-I}$$

is satisfied.

(b) *Suppose that $\mathbb{Q} := (\mathbb{Q}(B,C))_{B,C\in\mathscr{Y}}$ is a rate matrix satisfying (Q). If $\mathbb{Q}$ and A fulfill condition* (D-I)*, then the continuous conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ on $\mathscr{Y}$ which is associated to $\mathbb{Q}$ and the Markov semigroup $(T_t)_{t\geq 0}$ are in duality with respect to H. This is, the condition*

$$T_t H(\cdot,B)(\eta) = \mathbb{P}_t H(\eta,\cdot)(B), \quad \eta \in \mathscr{X},\ B \in \mathscr{Y}, \tag{D-S}$$

holds.

Proof.

(a) By the definition of the infinitesimal generator, the limit

$$\lim_{t\downarrow 0} \frac{1}{t}\big(T_t H(\cdot, B)(\eta) - H(\eta, B)\big) = AH(\cdot, B)(\eta)$$

exists for each $B \in \mathscr{Y}$ uniformly with respect to $\eta \in \mathscr{X}$. Hence it follows from (D-S) that

$$\lim_{t\downarrow 0} \frac{1}{t} \sum_{C\in\mathscr{Y}} \mathbb{P}_t(B, C)\big(H(\eta, C) - H(\eta, B)\big) = AH(\cdot, B)(\eta) \tag{3.4}$$

exists for each $B \in \mathscr{Y}$ uniformly with respect to $\eta \in \mathscr{X}$. Applying Lemma 3.1 with $h(\cdot) := H(\eta, \cdot) - H(\eta, B)$, where $B \in \mathscr{Y}$ is fixed, one obtains (D-I).

(b) If the condition (Q) holds for the rate matrix $\mathbb{Q}$, there exists a uniquely determined continuous conservative transition semigroup $(\mathbb{P}_t)_{t\geq 0}$ on $\mathscr{Y}$ with infinitesimal characteristics $\mathbb{Q}$. Let (D-I) be satisfied. Consider the function $v : [0, \infty) \times \mathscr{Y} \times \mathscr{X} \to \mathbb{R}$ defined by

$$v(t, V; \eta) := \mathbb{P}_t H(\eta, \cdot)(V), \quad t \geq 0,\ V \in \mathscr{Y},\ \eta \in \mathscr{X}.$$

Since $(\mathbb{P}_t)_{t\geq 0}$ is contractive and $H \in B(\mathscr{X} \times \mathscr{Y})$, one finds that

$$|v(t, V; \eta)| \leq \|H\|_\infty, \quad t \geq 0,\ V \in \mathscr{Y},\ \eta \in \mathscr{X},$$

therefore $v(\cdot, \cdot; \eta) \in B([0, \infty) \times \mathscr{Y})$ for each $\eta \in \mathscr{X}$. By Proposition 3.1, the function $v(\cdot, \cdot; \eta)$ is for fixed $\eta \in \mathscr{X}$ the unique solution in $B([0, \infty) \times \mathscr{Y})$ of the initial value problem

$$\begin{aligned} \frac{d}{dt} v(t, V) &= \sum_{C\in\mathscr{Y}} \mathbb{Q}(V, C)[v(t, C) - v(t, V)], \quad V, C \in \mathscr{Y},\ t > 0, \\ v(0+; V) &= H(\eta, V), \quad V \in \mathscr{Y}. \end{aligned} \tag{3.5}$$

Define a function $u : [0, \infty) \times \mathscr{X} \times \mathscr{Y} \to \mathbb{R}$ by

$$u(t, \eta; B) := T_t H(\cdot, B)(\eta), \quad t \geq 0,\ \eta \in \mathscr{X},\ B \in \mathscr{Y}.$$

Since $(T_t)_{t\geq 0}$ is contractive and $H \in B(\mathscr{X} \times \mathscr{Y})$, one finds that

$$|u(t, \eta; B)| \leq \|H\|_\infty, \quad t \geq 0,\ \eta \in \mathscr{X},\ B \in \mathscr{Y},$$

therefore $u(\cdot, \cdot; \eta) \in B([0, \infty) \times \mathscr{Y})$ for each $\eta \in \mathscr{X}$. Consider $u(\cdot, \cdot; B)$ for fixed $B \in \mathscr{Y}$. For the Markov semigroup $(T_t)_{t\geq 0}$ and its infinitesimal generator A the following equation is valid on $\vartheta(A)$

$$\frac{d}{dt} T_t = T_t A, \quad t \geq 0,$$

see, for instance, [5]. Since $H(\cdot, B) \in \vartheta(A)$, this yields

$$\frac{d}{dt}u(t,\eta;B) = T_t AH(\cdot,B)(\eta), \quad t \geq 0,\ \eta \in \mathscr{X},\ B \in \mathscr{Y}.$$

It follows from (D-I) that, for each $B \in Y$, the function $\eta \mapsto \mathbb{Q}H(\eta,\cdot)(B)$ is continuous on $\mathscr{X}$, because $AH(\cdot,B) \in C(\mathscr{X})$. Consequently, applying (D-I) to the above equation yields

$$\frac{d}{dt}u(t,\eta;B) = T_t\Big(\sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[H(\cdot,C)-H(\cdot,B)]\Big)(\eta),\ \ t \geq 0,\ \eta \in \mathscr{X},\ B \in \mathscr{Y}.$$

Now choose some sequence $(\mathscr{Y}_n)_{n\in\mathbb{N}}$, the $\mathscr{Y}_n$'s being finite subsets of $\mathscr{Y}$ which satisfy $\mathscr{Y}_n \uparrow \mathscr{Y}$ for $n \to \infty$. As T_t is bounded, one finds

$$\begin{aligned}
\frac{d}{dt}u(t,\eta;B) &= T_t\Big(\sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[H(\cdot,C) - H(\cdot,B)]\Big)(\eta)\\
&= T_t\Big(\lim_{n\to\infty}\sum_{C\in\mathscr{Y}_n} \mathbb{Q}(B,C)[H(\cdot,C) - H(\cdot,B)]\Big)(\eta)\\
&= \lim_{n\to\infty}\sum_{C\in\mathscr{Y}_n} \mathbb{Q}(B,C)[T_tH(\cdot,C)(\eta) - H(\eta,B)]\\
&= \lim_{n\to\infty}\sum_{C\in\mathscr{Y}_n} \mathbb{Q}(B,C)[u(t,\eta;C) - u(t,\eta;B)]\\
&= \sum_{C\in\mathscr{Y}} \mathbb{Q}(B,C)[u(t,\eta;C) - u(t,\eta;B)], \quad t \geq 0,\ \eta \in \mathscr{X}.
\end{aligned}$$

Observe further that $u(0+,\eta;B) = H(\eta,B)$, $\eta \in \mathscr{X}$. Since $B \in \mathscr{Y}$ can be chosen arbitrarily, we obtain that the function $u(\cdot,\eta;\cdot)$ satisfies the initial value problem (IVP) for each $\eta \in \mathscr{X}$. As the solution of this initial value problem is unique in $B([0,\infty) \times \mathscr{Y})$, it follows that

$$v(t,B;\eta) = u(t,\eta;B), \quad t \geq 0,\ \eta \in \mathscr{X},\ B \in \mathscr{Y}.$$

This is (D-S). □

Remark 3.2. Actually, it is enough to require in Theorem 3.1 (b) that the rate matrix $\mathbb{Q}$ and the semigroup $(\mathbb{P}_t)_{t\geq 0}$ generated by $\mathbb{Q}$ satisfy condition (PQ). However, if only $\mathbb{Q}$ is given explicitly, it is often easier to verify that the condition (Q) is met.

Corollary 3.1. *Suppose that $(A_i, \vartheta(A_i))$, $i \in I$, are Markov generators on $\mathscr{X}$, where I is a countable index set. Assume that there is a set $D \subset \cap_{i\in I}\vartheta(A_i)$*

with $\mathbb{1} \in D$ *which is a core*[4] *for each operator* A_i, $i \in I$, *such that the operator*

$$A : D \longrightarrow C(\mathscr{X}) : Af := \sum_{i \in I} A_i f^5$$

is a Markov pregenerator whose closure is a Markov generator.

Let H *be a real-valued bounded measurable function on* $\mathscr{X} \times \mathscr{Y}$ *satisfying the condition*

$$H(\cdot, B) \in D, \quad B \in \mathscr{Y}.$$

Assume further that there are Markov chains with infinitesimal characteristics $\mathbb{Q}_i$ *which are* H*-dual to the Markov processes generated by* A_i, $i \in I$. *If*

$$-\infty < \mathbb{Q}(B, C) := \sum_{i \in I} \mathbb{Q}_i(B; C) < \infty, \quad B, C \in \mathscr{Y}, \tag{3.6}$$

and the corresponding Q-matrix[6] $\mathbb{Q} = (\mathbb{Q}(B, C)_{B,C \in \mathscr{Y}}$ *satisfies* (Q), *then the Markov process corresponding to* A *is* H*-dual to the Markov chain on* $\mathscr{Y}$ *with infinitesimal characteristics* $\mathbb{Q}$.

Remark 3.3.

(1) Since each (A_i, D), $i \in I$ is a Markov pregenerator, the operator $A : D \to C(\mathscr{X}) : Af := \sum_{i \in I} A_i f$ satisfies the conditions (G1) and (G2) of § 2.1 and is therefore a Markov pregenerator. It is not immediate, in general, that (A, D) satisfies the condition (G3), as well.

(2) Suppose that A_1 and A_2 are generators of IPS as defined in § 2.2 with corresponding families of transition rates $(c_T^{(1)}(\cdot, \cdot))_{T \in \mathscr{T}}$ and $(c_T^{(2)}(\cdot, \cdot))_{T \in \mathscr{T}}$, respectively. Then $D(\mathscr{X}) \subset \vartheta(A_1) \cap \vartheta(A_2)$ is a core for A_i, $i = 1, 2$ and the operator $(A, D(\mathscr{X}))$ is the pregenerator of an IPS with transition family $(c_T(\cdot, \cdot))_{T \in T}$ satisfying $c_T(\cdot, \cdot) = c_T^{(1)}(\cdot, \cdot) + c_T^{(2)}(\cdot, \cdot)$. It is easily checked that the family $(c_T(\cdot, \cdot))_{T \in T}$ is admissible, if $(c_T^{(1)}(\cdot, \cdot))_{T \in \mathscr{T}}$ and $(c_T^{(2)}(\cdot, \cdot))_{T \in \mathscr{T}}$ are admissible. Thus one obtains by following the arguments in § 2.2 that the closure of $(A, D(\mathscr{X}))$ is a Markov generator.

(3) Suppose that A is the generator of an IPS as defined in § 2.2 with corresponding family of transition rates $(c_T(\cdot, \cdot))_{T \in \mathscr{T}}$. According to (2.1), it holds that $Af = \sum_{T \in \mathscr{T}} A_T f$, $f \in D(\mathscr{X})$, where $A_T : C(\mathscr{X}) \to C(\mathscr{X})$,

[4] See [5, §1.3] for the definition of a core.

[5] The convergence is with respect to the supremum norm in $C(\mathscr{X})$.

[6] $\mathbb{Q}(B) := \sum_{C \neq B} \mathbb{Q}(B, C) = \sum_{C \neq B} \sum_{i \in I} \mathbb{Q}_i(B, C) = \sum_{i \in I} \sum_{C \neq B} \mathbb{Q}_i(B, C) = \sum_{i \in I} -\mathbb{Q}_i(B, B) = -\mathbb{Q}(B, B) < \infty$, $B \in \mathscr{Y}$, where the rearrangement in the order of summation is allowed because all summands are non-negative.

$T \in \mathscr{T}$ are bounded Markov generators that describe the local transitions. Thus $D(\mathscr{X}) \subset \bigcup_{T\in\mathscr{T}} \vartheta(A_T)$ is a core for each A_T, $T \in \mathscr{T}$, and the operator $(A, D(\mathscr{X}))$ is a pregenerator of a Markov process. If the local mechanisms A_T are H-dual to Markov chains on $\mathscr{Y}$ with infinitesimal characteristics $\mathbb{Q}_T$, $T \in \mathscr{T}$, then the H-dual of A exists and has infinitesimal characteristics $\mathbb{Q} = \sum_{T\in\mathscr{T}} \mathbb{Q}_T$, if $\mathbb{Q}$ is well-defined in the sense of (3.6) and satisfies condition (Q).

Proof of Corollary 3.1. By Theorem 3.1, the infinitesimal characteristics $\mathbb{Q}_i$ of the H-dual chains satisfy

$$A_i H(\cdot, B)(\eta) = \mathbb{Q}_i H(\eta, \cdot)(B), \quad i \in I,\ \eta \in \mathscr{X},\ B \in \mathscr{Y}.$$

Hence

$$\begin{aligned} AH(\cdot, B)(\eta) &= \sum_{i\in I} A_i H(\cdot, B)(\eta) = \sum_{i\in I} \mathbb{Q}_i H(\eta, \cdot)(B) \\ &= \sum_{i\in I} \sum_{C\neq B} \mathbb{Q}_i(B, C)\big(H(\eta, C) - H(\eta, B)\big) \\ &= \sum_{C\neq B} \sum_{i\in I} \mathbb{Q}_i(B, C)\big(H(\eta, C) - H(\eta, B)\big) \\ &= \mathbb{Q}H(\eta, \cdot)(B), \quad \eta \in \mathscr{X},\ B \in \mathscr{Y}. \end{aligned}$$

The change in the order of summation is allowed since the series is absolutely convergent. Indeed,

$$\begin{aligned} &\sum_{i\in I} \sum_{C\neq B} \big|\mathbb{Q}_i(B, C)\big(H(\eta, C) - H(\eta, B)\big)\big| \\ &\quad \leq \sum_{i\in I} \sum_{C\neq B} \mathbb{Q}_i(B, C) \underbrace{|H(\eta, C) - H(\eta, B)|}_{\leq 2} \leq 2 \sum_{i\in I} \mathbb{Q}_i(B) < \infty. \end{aligned}$$

Since the matrix $\mathbb{Q}$ satisfies (Q), there exists a uniquely determined continuous and conservative transition semigroup on $\mathscr{Y}$ with infinitesimal characteristics $\mathbb{Q}$. By Theorem 3.1, the corresponding Markov chain is H-dual to the IPS generated by A. □

4. Duals of IPS

Now the results of Section 3 are applied to special IPS and special duality functions H. Thereby it is illustrated that the criterion for duality which has been developed above is indeed manageable for a variety of different interacting particle systems. Note that the duality relations that will be derived in the examples below have been asserted in the literature already. However the proof of these relations was either only scheduled or even omitted in the original publication.

In § 4.1 to § 4.3, IPS with state space $\mathscr{X} = W^S$ are considered, where $W = \{0,1\}$ and $S = \mathbb{Z}^d$ with $d \in \mathbb{N}$. Define

$$\mathscr{T}_\emptyset := \{T \subset S : |T| < \infty\}, \quad \mathscr{T} := \mathscr{T}_\emptyset \backslash \{\emptyset\}.$$

For the duality function the following map $H : \mathscr{X} \times \mathscr{T}_\emptyset \to \{0,1\}$ is chosen

$$H(\eta, B) := \prod_{x \in B} \eta(x), \quad \eta \in \mathscr{X},\ B \in \mathscr{T}, \quad H(\cdot, \emptyset) \equiv 1. \tag{4.1}$$

Hence the state space of H-dual Markov chains is $\mathscr{Y} := \mathscr{T}_\emptyset$. Obviously, $H(\cdot, B) \in T(\mathscr{X}) \subseteq D(\mathscr{X})$ for each $B \in \mathscr{T}_\emptyset$. Note that

$$H(\eta, B) = \prod_{x \in B} \eta(x) = 1 \iff \eta(x) = 1, \quad x \in B.$$

In § 4.4–§ 4.5, the local state space W as well as the duality function H are chosen to be more general.

4.1. Spin-flip systems

Spin-flip systems are IPS with single site space $W = \{0,1\}$ and admissible transition rates $c = (c_T(\cdot,\cdot))_{T \in \mathscr{T}}$ satisfying $c_T(\eta, \{v\}) = 0$, $T \in \mathscr{T}$, $\eta \in \mathscr{X}$, $v \in \mathscr{X}_T$ unless $T = \{x\}$ for some $x \in S$. For convenience, denote for $\eta \in \mathscr{X}$, $x \in S$,

$$\eta^x(z) := \begin{cases} \eta(z), & z \neq x, \\ 1 - \eta(x), & z = x, \end{cases} \quad z \in S,$$

and

$$c(x, \eta) := c_{\{x\}}\big(\eta, \{1 - \eta(x)\}\big).$$

The corresponding Markov pregenerator $A : D(\mathscr{X}) \to C(\mathscr{X})$ takes the form

$$Af(\eta) = \sum_{x \in S} c(x, \eta)(f(\eta^x) - f(\eta)), \quad f \in D(\mathscr{X}),\ \eta \in \mathscr{X}.$$

As above, the Markov semigroup that is generated by A is denoted by $(T_t)_{t \geq 0}$.

The results of Section 3 shall be applied to spin-flip systems where the transition rates have a special structure, compare [9, § III.4]. In detail, let $p : S \times \mathscr{T}_\emptyset \to [0,1]$ be a map that satisfies

$$\sum_{F \in \mathscr{T}_\emptyset} p(x, F) = 1, \quad x \in S, \tag{4.2}$$

$$\sup_{x \in S} \sum_{F \in \mathscr{T}_\emptyset} p(x, F)|F| =: \kappa < \infty.$$

It is supposed that the transition rates c of the spin-flip system take the form

$$c(x,\eta) := \eta(x) + (1-2\eta(x)) \sum_{F\in\mathscr{T}_\emptyset} p(x,F)H(\eta,F), \quad \eta\in\mathscr{X},\ x\in S. \tag{4.3}$$

Note that (4.2) guarantees that the family $c = (c(x,\cdot))_{x\in S}$ is admissible [9, § III.4]. A rate matrix $\mathbb{Q}$ on $\mathscr{T}_\emptyset$ is specified via[7,8]

$$\mathbb{Q}(B,C) := \sum_{x\in B}\sum_{F\in\mathscr{T}_\emptyset} p(x,F)\delta_{F\cup(B\setminus\{x\})}(C), \quad B\in\mathscr{T},\ C\in\mathscr{T}_\emptyset,\ B\neq C, \tag{4.4}$$

and

$$\mathbb{Q}(B,B) := -\mathbb{Q}(B) := -\sum_{C\in\mathscr{T}_\emptyset\setminus\{B\}} \mathbb{Q}(B,C), \quad B\in\mathscr{T}_\emptyset.$$

One finds

$$\begin{aligned}\mathbb{Q}(B) &= \sum_{C\in\mathscr{T}_\emptyset, C\neq B} \mathbb{Q}(B,C) = \sum_{C\in\mathscr{T}_\emptyset, C\neq B}\ \sum_{x\in B}\sum_{F\in\mathscr{T}_\emptyset} p(x,F)\delta_{F\cup(B\setminus\{x\})}(C)\\ &\le \sum_{x\in B}\sum_{F\in\mathscr{T}_\emptyset} p(x,F) = |B| < \infty, \quad B\in\mathscr{T}_\emptyset,\end{aligned}$$

thus the matrix $\mathbb{Q}$ is actually well-defined. In addition, following [9, § III.4], one obtains

$$\begin{aligned}&\sum_{C\in\mathscr{T}_\emptyset, C\neq B} \mathbb{Q}(B,C)[|C|-|B|]\\ &\qquad = \sum_{x\in B}\sum_{F\in\mathscr{T}_\emptyset} p(x,F)\big[|(B\setminus\{x\})\cup F| - |B|\big]\\ &\qquad \le \sum_{x\in B}\sum_{F\in\mathscr{T}_\emptyset} p(x,F)(|F|-1) \le \kappa|B|, \quad B\in\mathscr{T}_\emptyset.\end{aligned}$$

Choosing $\mathscr{T}_n := \{T\in\mathscr{T}_\emptyset : |T|\le n\}$, $n\in\mathbb{N}$, and $\varphi(B) := |B|$, $B\in\mathscr{T}_\emptyset$, the condition (Q) of § 2.3 is satisfied. Hence the matrix $\mathbb{Q}$ generates a uniquely determined continuous conservative transition semigroup $(\mathbb{P}_t)_{t\ge 0}$ on $\mathscr{T}_\emptyset$. One easily verifies that A and $\mathbb{Q}$ satisfy condition (D-I). Indeed, with

$$H(\eta^x,B) - H(\eta,B) = \begin{cases}(1-2\eta(x))H(\eta,B\setminus\{x\}) & \text{for } x\in B,\\ 0 & \text{for } x\notin B,\end{cases}$$

and

$$H(\eta,C)H(\eta,D) = H(\eta,C\cup D), \quad C,D\in\mathscr{T}_\emptyset,$$

[7] The Kronecker symbol satisfies $\delta_B(C) = 1$ if $C = B$ and $\delta_B(C) = 0$ otherwise.
[8] By convention, a sum taken over the empty set is zero.

it follows from (4.3) that

$$
\begin{aligned}
&A\,H(\cdot,B)(\eta)\\
&\quad=\sum_{x\in B} c(x,\eta)[H(\eta^x,B)-H(\eta,B)]=\sum_{x\in B} c(x,\eta)(1-2\eta(x))H(\eta,B\setminus\{x\})\\
&\quad=\sum_{x\in B}\Big\{\Big(\eta(x)+(1-2\eta(x))\sum_{F\in\mathscr{T}_\emptyset} p(x,F)H(\eta,F)\Big)(1-2\eta(x))H(\eta,B\setminus\{x\})\Big\}\\
&\quad=\sum_{x\in B}\eta(x)(1-2\eta(x))H(\eta,B\setminus\{x\})\\
&\qquad+\sum_{x\in B}(1-2\eta(x))^2\sum_{F\in\mathscr{T}_\emptyset} p(x,F)H(\eta,F)H(\eta,B\setminus\{x\})\\
&\quad=\sum_{x\in B}\sum_{F\in\mathscr{T}_\emptyset} p(x,F)\big[H(\eta,F\cup(B\setminus\{x\})-H(\eta,B)\big]\\
&\quad=\sum_{C\in\mathscr{T}_\emptyset}\mathbb{Q}(B,C)[H(\eta,C)-H(\eta,B)],\quad \eta\in\mathscr{X},\ B\in\mathscr{T}_\emptyset.
\end{aligned}
$$

Hence the IPS generated by A and the Markov chain corresponding to $\mathbb{Q}$ are in duality w.r.t. H.

Example 4.1. If one specializes

$$
p(x,F):=\begin{cases}0, & \text{if } |F|\neq 1\\ p^v(x,y), & \text{if } F=\{y\}\subset S,\end{cases}
$$

where p^v is an irreducible stochastic matrix on S, then the rates in (4.3) take the form

$$
c^v(x,\eta)=\sum_{y\in S} p^v(x,y)\big(\eta(y)-\eta(x)\big)^2,\quad x\in S,\ \eta\in\mathscr{X}. \tag{4.5}
$$

These are the transition rates of a so-called *linear voter system.* The infinitesimal characteristics of its H-dual Markov chain on $\mathscr{T}_\emptyset$ are given by

$$
\mathbb{Q}^v(B,C)=\sum_{x\in B}\sum_{y\in C} p(x,y)\delta_{\{y\}\cup(B\setminus\{x\})}(C),\quad B\in\mathscr{T},\ C\in\mathscr{T}_\emptyset,\ B\neq C. \tag{4.6}
$$

4.2. Symmetric exclusion processes

Symmetric exclusion processes fall into the class of *spin-exchange processes.* The latter processes are IPS with $W=\{0,1\}$ and admissible transition rates $c=(c_T(\cdot,\cdot))_{T\in\mathscr{T}}$ satisfying $c_T(\eta,\{v\})=0$, $T\in\mathscr{T}$, $\eta\in\mathscr{X}$, $v\in\mathscr{X}_T$ unless

$T = \{x, y\}$ for some $x, y \in S$, $x \neq y$ and $v(x) = \eta(y)$, $v(y) = \eta(x)$. For convenience, denote for $\eta \in \mathscr{X}$, $x, y \in S$, $x \neq y$,

$$\eta^{xy}(z) := \begin{cases} \eta(z), & z \neq x, y, \\ \eta(y), & z = x, \\ \eta(x), & z = y, \end{cases} \qquad z \in S,$$

and

$$c(x, y, \eta) := c_{\{x,y\}}(\eta, \{v\}), \quad \text{where } v \in \mathscr{X}_{\{x,y\}} \text{ with } v(x) := \eta(y),\ v(y) := \eta(x).$$

If the transition rates $c(x, y, \eta)$ admit a representation $c(x, y, \eta) = c^e(x, y, \eta)$ with

$$c^e(x, y, \eta) = p^e(x, y)\eta(x)(1 - \eta(y)), \quad x, y \in S,\ x \neq y,\ \eta \in \mathscr{X}, \tag{4.7}$$

where $p^e = (p^e(x, y))_{x,y \in S}$ is an irreducible and symmetric stochastic matrix on S with $p^e(x, x) = 0$, $x \in S$, then the corresponding IPS is a *symmetric exclusion process*. Note that the family of transition rates $(c(x, y, \cdot))_{x,y \in S}$ is admissible, since, by the symmetry of p^e, the condition $\sup_{y \in S} \sum_{x \in S} p^e(x, y) < \infty$ is satisfied [9, Ch. VIII]. The corresponding Markov pregenerator $A^e : D(\mathscr{X}) \to C(\mathscr{X})$ takes the form

$$A^e f(\eta) = \sum_{x,y \in S} p(x, y)\eta(x)(1 - \eta(y))(f(\eta^{xy}) - f(\eta)), \quad f \in D(\mathscr{X}),\ \eta \in \mathscr{X}. \tag{4.8}$$

Let

$$\begin{aligned} \mathbb{Q}^e(B, C) &:= \sum_{x \in B} \sum_{y \in C \setminus B} p^e(x, y)\delta_{(B \setminus \{x\}) \cup \{y\}}(C), \quad B, C \in \mathscr{T}_\emptyset,\ B \neq C, \\ \mathbb{Q}^e(B, B) &:= -\sum_{C \in \mathscr{T}_\emptyset \setminus \{B\}} \mathbb{Q}^e(B, C), \quad B \in \mathscr{T}_\emptyset. \end{aligned} \tag{4.9}$$

Then one has for $B \in \mathscr{T}_\emptyset$, $C \in \mathscr{T}_\emptyset$, $B \neq C$, that

$$\mathbb{Q}^e(B, C) \leq \sum_{x \in B} \sum_{y \in C} p^e(x, y)\delta_{\{y\} \cup (B \setminus \{x\})}(C) = \mathbb{Q}^v(B, C)$$

and

$$\mathbb{Q}^e(B) \leq \mathbb{Q}^v(B),$$

where $\mathbb{Q}^v$ is the rate matrix of a linear voter system, see (4.6). This means that $\mathbb{Q}^e$ is a well-defined rate matrix which satisfies condition (Q2) for $\varphi(B) = |B|$, $B \in \mathscr{T}_\emptyset$. It satisfies the condition (Q1), as well, since one finds that

$$\delta_{(B \setminus \{x\}) \cup \{y\}}(C)[|C| - |B|] = 0 \leq |B|,$$

for $B, C \in \mathscr{T}_\emptyset$ and $x, y \in S$, $x \in B, y \notin B$. Now it shall be verified that A^e and $\mathbb{Q}^e$ are H-dual, where H is the defined in (4.1). Using that

$$H(\eta^{xy}, B) = H\big(\eta, (B \setminus \{x\}) \cup \{y\}\big) \quad \text{for } x \in B,\ y \notin B,$$

and applying the symmetry of p^e, one finds, following [9, § VIII.1],

$$\begin{aligned} A^e H(\cdot, B)(\eta) &= \sum_{x \in S} \sum_{y \in S} p^e(x,y)\eta(x)(1-\eta(y))[H(\eta^{xy}, B) - H(\eta, B)] \\ &= \sum_{x \in B} \sum_{y \notin B} p^e(x,y)\big[\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))\big] \\ &\quad \times \big[H(\eta, (B \setminus \{x\}) \cup \{y\}) - H(\eta, B)\big] \\ &= \sum_{x \in B} \sum_{y \notin B} p^e(x,y)\big[H(\eta, (B \setminus \{x\}) \cup \{y\}) - H(\eta, B)\big] \\ &= \sum_{C \in \mathscr{T}_\emptyset} \mathbb{Q}^e(B, C)[H(\eta, C) - H(\eta, B)], \quad \eta \in \mathscr{X},\ B \in \mathscr{T}_\emptyset. \end{aligned}$$

The third equality follows from the fact that $\big[\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))\big] = 1$ if $\eta(x) \neq \eta(y)$. Thus A^e and $\mathbb{Q}^e$ satisfy (D-I) w.r.t. H. Consequently, by Theorem 3.1, the symmetric exclusion process has an H-dual Markov chain on $\mathscr{T}_\emptyset$ with infinitesimal characteristics $\mathbb{Q}^e$.

4.3. Voter-exclusion processes

Suppose that $(c^v(x, \cdot))_{x \in S}$ and $(c^e(x, y, \cdot))_{x,y \in S}$ are the transition rates of a linear voter system w.r.t. the irreducible stochastic matrix p^v on S and of a spin-exchange process w.r.t. the symmetric and irreducible stochastic matrix p^e on S, respectively, as defined in (4.5) and (4.7). The corresponding Markov pregenerators are given by

$$A^v f(\eta) = \sum_{x \in S} \sum_{y \in S} p^v(x,y)(\eta(y) - \eta(x))^2 (f(\eta^x) - f(\eta)), \quad \eta \in \mathscr{X},\ f \in D(\mathscr{X}),$$

and

$$A^e f(\eta) = \sum_{x \in S} \sum_{y \notin S} p^e(x,y)\eta(x)(1-\eta(y))(f(\eta^{xy}) - f(\eta)), \quad \eta \in \mathscr{X},\ f \in D(\mathscr{X}).$$

The IPS corresponding to $A := A^v + A^e$ on $D(\mathscr{X})$ is called *voter-exclusion process*[9].

Let $\mathbb{Q}^v$ and $\mathbb{Q}^e$ be the infinitesimal characteristics of the H-dual chains corresponding to A^v and A^e, respectively, as given in (4.6) and (4.9). Both

[9] see [7].

$A^v, \mathbb{Q}^v$ and A^e, $\mathbb{Q}^e$ satisfy (D-I) with respect to the duality function H given in (4.1). Since $\mathbb{Q}^v$ and $\mathbb{Q}^e$ meet condition (Q) w.r.t. the same function $\varphi = |\cdot|$, one finds that $\mathbb{Q} := \mathbb{Q}^v + \mathbb{Q}^e$ satisfies (Q). Thus Corollary 3.1 applies. It follows that A and $\mathbb{Q}$ satisfy (D-I) with respect to H. Hence the Markov chain which is H-dual to the voter-exclusion process has the infinitesimal characteristics

$$\mathbb{Q}(B,C) = \sum_{x\in B}\sum_{y\in C} \big(p^v(x,y) + p^e(x,y)\mathbb{1}_{B^c}(y)\big)\delta_{\{(B\setminus\{x\})\cup\{y\}}(C),$$

for $B \in \mathscr{T}_\emptyset$, $C \in \mathscr{T}_\emptyset$, $B \neq C$.

4.4. Symmetric two-particle exclusion-eating process

The symmetric two-particle exclusion-eating process was considered in [10]. It is an IPS on $\mathscr{X} = W^S$ with $W := \{0,1,2\}$ and S a countable set, where only transitions $\eta \to \eta_{x\to y}$, $x, y \in S$, $x \neq y$, have positive rates. These transitions are defined for $x, y \in S$, $x \neq y$ by

$$\eta_{x\to y}(z) := \begin{cases} \eta(z), & \text{if } z \notin \{x,y\}, \\ \eta(x), & \text{if } z = y, \\ \min\{2\eta(y), \eta(x)\}, & \text{if } z = x, \end{cases} \qquad z \in S.$$

Suppose that p is a symmetric irreducible stochastic matrix on S. Then the generator of the two-particle exclusion-eating process is given by the closure of the operator

$$Af(\eta) = \sum_{x\in S}\sum_{y\in S} \big(1 - \delta_0(\eta(x))\big)\big(1 - \delta_{\eta(x)}(\eta(y))\big)p(x,y)(f(\eta_{x\to y}) - f(\eta)),$$

where $f \in D(\mathscr{X})$, $\eta \in \mathscr{X}$. For $\eta \in \mathscr{X}$ and $x, y \in S$, $x \neq y$ with $\eta(x) \neq 0$, $\eta(y) \neq \eta(x)$, it holds that

$$\eta_{x\to y} = \begin{cases} \eta^{xy}, & \text{if } \eta(y) = 0, \\ \eta^x, & \text{if } \eta(y) \neq 0, \end{cases}$$

with exclusion transformation

$$\eta^{xy}(z) := \begin{cases} \eta(z), & \text{if } z \notin \{x,y\}, \\ \eta(x), & \text{if } z = y, \\ \eta(y), & \text{if } z = x, \end{cases} \qquad z \in S,$$

and spin-flip transformation

$$\eta^x(z) := \begin{cases} \eta(z), & \text{if } z \neq x, \\ 3 - \eta(x), & \text{if } z = x, \end{cases} \qquad z \in S.$$

Consequently, the operator A can be decomposed into the sum $A = A_1 + A_2$, where

$$A_1 f(\eta) = \sum_{x\in S}\sum_{y\in S} \big(1 - \delta_0(\eta(x))\big)\delta_0(\eta(y))p(x,y)(f(\eta^{xy}) - f(\eta))$$

and

$$A_2 f(\eta) = \sum_{x\in S}\sum_{y\in S} \delta_0(\eta(x)\eta(y))p(x,y)(f(\eta^{x}) - f(\eta))$$

for $\eta \in \mathscr{X}$, $f \in D(\mathscr{X})$. It is easily verified that both A_1 and A_2 are IPS-pregenerators of the form (2.1), each one with admissible transition rates. Therefore, compare Remark 3.3 (2), the closure of A is indeed a Markov generator in $C(\mathscr{X})$.

Now choose $\mathscr{Y} := \{\mathbf{C} = (C_1, C_2) \in \mathscr{T}_\emptyset \times \mathscr{T}_\emptyset : C_1 \subseteq C_2\}$ to be the state space of the dual Markov chain. The duality function $H|\mathscr{X} \times \mathscr{Y} \to \{0,1\}$ shall be given by

$$H(\eta, \mathbf{B}) := \prod_{x\in B_1} \delta_2(\eta(x)) \prod_{y\in B_2} (1 - \delta_0(\eta(y)), \quad \eta \in \mathscr{X},\ \mathbf{B} = (B_1, B_2) \in \mathscr{Y}. \tag{4.10}$$

Obviously, $H(\cdot, \mathbf{B}) \in T(\mathscr{X})$, $\mathbf{B} \in \mathscr{Y}$. Two rate matrices $\mathbb{Q}_1$, $\mathbb{Q}_2$ on $\mathscr{Y}$ are defined by

$$\mathbb{Q}_1(\mathbf{B}, \mathbf{C}) := \sum_{x\in B_1}\sum_{y\notin B_2} p(x,y)\delta_{B_1\setminus\{x\}\cup\{y\}}(C_1)\delta_{B_2\setminus\{x\}\cup\{y\}}(C_2)$$

and

$$\mathbb{Q}_2(\mathbf{B}, \mathbf{C}) := \sum_{x\in B_2\setminus B_1}\sum_{y\notin B_2} p(x,y)\delta_{B_1}(C_1)\delta_{B_2\setminus\{x\}\cup\{y\}}(C_2),$$

where $\mathbf{B} = (B_1, B_2)$, $\mathbf{C} = (C_1, C_2) \in \mathscr{Y}$, $\mathbf{B} \neq \mathbf{C}$.

From [10, Lemma 2.1] one has that (D_0) holds for a symmetric two-particle exclusion-eating process and a Markov chain on $\mathscr{Y}$ having the infinitesimal characteristics $\mathbb{Q} := \mathbb{Q}_1 + \mathbb{Q}_2$, that is

$$(A_1 + A_2)H(\cdot, \mathbf{B})(\eta) = \sum_{\mathbf{C}\in\mathscr{Y}} (\mathbb{Q}_1 + \mathbb{Q}_2)(\mathbf{B}, \mathbf{C})[H(\eta, \mathbf{C}) - H(\eta, \mathbf{B})],$$

for $\eta \in \mathscr{X}$, $\mathbf{B}, \mathbf{C} \in \mathscr{Y}$. To conclude that the semigroup duality relation (D-S) holds, one has to show that $\mathbb{Q}$ satisfies the condition (Q). To prove this, consider $\mathbb{Q}_1$ and $\mathbb{Q}_2$ separately. With $\mathbb{Q}_i(\mathbf{B}) := \sum_{\mathbf{C}\neq\mathbf{B}} \mathbb{Q}(\mathbf{B}, \mathbf{C})$, $\mathbf{B} \in \mathscr{Y}$, $i = 1, 2$, one finds that

$$\mathbb{Q}_1(\mathbf{B}) \leq \sum_{x\in B_1}\sum_{y\notin B_2} p(x,y) \leq |B_1|,$$

and

$$\mathbb{Q}_2(\mathbf{B}) \le \sum_{x\in B_2\setminus B_1} \sum_{y\notin B_2} p(x,y) \le |B_2| - |B_1|,$$

for $\mathbf{B} = (B_1, B_2) \in \mathscr{Y}$. Therefore $\mathbb{Q}(\mathbf{B}) = \mathbb{Q}_1(\mathbf{B}) + \mathbb{Q}_2(\mathbf{B}) \le |B_2|$, $\mathbf{B} = (B_1, B_2) \in \mathscr{Y}$. Because of $|B \setminus \{x\} \cup \{y\}| = |B|$ for $B \in \mathscr{T}_\emptyset$, $x \in B$, $y \notin B$, it holds for $i = 1, 2$ that $\mathbb{Q}_i(\mathbf{B}, \mathbf{C}) = 0$ for $\mathbf{B} = (B_1, B_2)$, $\mathbf{C} = (C_1, C_2) \in \mathscr{Y}$ with $|C_2| \neq |B_2|$. Hence

$$\sum_{\mathbf{C}\in\mathscr{Y}} \mathbb{Q}(\mathbf{B},\mathbf{C})(|C_2| - |B_2|) = 0, \quad \mathbf{B} \in \mathscr{Y}.$$

Choosing now $\mathscr{Y}_n := \{\mathbf{C} = (C_1, C_2) \in \mathscr{Y} : |C_2| \le n\}$, $n \in \mathbb{N}$, and $\varphi : \mathscr{Y} \to [0,\infty) : \varphi(\mathbf{B}) := |B_2|$, $\mathbf{B} = (B_1, B_2) \in \mathscr{Y}$, one finds that (Q) is satisfied.

This shows that the Markov chain generated by $\mathbb{Q}$ is H-dual to the symmetric two-particle exclusion-eating process. These arguments complete the proof of [10, Theorem 2.7].

4.5. Lattice gas with energy

In [12], Nagahata studies IPS in $\mathscr{X} := W^{\mathbb{Z}^d}$, $W := \{0, 1, \ldots, M\}$, $M, d \in \mathbb{N}$, where transitions $\eta \to \eta^{x\to y}$ and $\eta \to \eta^{x,y}$ are considered that are given by

$$\eta^{x\to y}(z) := \begin{cases} \eta(x) - 1, & z = x, \\ \eta(y) + 1, & z = y, \\ \eta(z), & z \notin \{x, y\}, \end{cases}$$

and

$$\eta^{x,y}(z) := \begin{cases} \eta(y), & z = x, \\ \eta(x), & z = y, \\ \eta(z), & z \notin \{x, y\}, \end{cases}$$

$x, y \in \mathbb{Z}^d$, $x \neq y$. These transitions shall occur with rates $C_{ge}(\eta(x))$ and $C_{ex}(\eta(x))$, if x and y are neighboring lattice sites, that is $|x - y| = 1$[10]. The transition rates $C_{ge}, C_{ex} : \mathbb{Z}^d \to [0,\infty)$ are given by

$$C_{ge}(k) := c_{ge}(k)\mathbb{1}_{[2,M]}(k), \quad C_{ex}(k) := c_{ex}(k)(1 - \delta_0(k)), \quad k \in W,$$

where $c_{ge}, c_{ex} : W \to (0,\infty)$ are specified in advance.

It is easily checked that the linear operators $A_{ge}, A_{ex} : \mathrm{T}(\mathscr{X}) \to C(\mathscr{X})$ defined by

$$A_{ge}f(\eta) := \sum_{x,y\in\mathbb{Z}^d} \delta_1(|x-y|)C_{ge}(\eta(x))(\eta(x))\mathbb{1}_{[1,M-1]}(\eta(y))(f(\eta^{x\to y}) - f(\eta)),$$

[10]Here $|x| := \sqrt{\sum_{n=1}^d x_n^2}$, $x \in \mathbb{Z}^d$.

and

$$A_{ex}f(\eta) := \sum_{x,y\in\mathbb{Z}^d} \delta_1(|x-y|)C_{ex}(\eta(x))\delta_0(\eta(y))(f(\eta^{x,y}) - f(\eta)),$$

$f \in \mathrm{T}(\mathscr{X})$, $\eta \in \mathscr{X}$, as well as the sum $A := A_{ge} + A_{ex}$ are Markov pregenerators in $C(\mathscr{X})$ whose closures in $C(\mathscr{X})$ are Markov generators.

According to Nagahata [12], the associated Feller process is called a *lattice gas with energy*. It is thought as a model of the time evolution of a gas consisting of particles with several energy levels. For $\eta \in \mathscr{X}, x \in \mathbb{Z}^d$, $\eta(x) = 0$ is interpreted as 'site x is vacant' and $\eta(x) \neq 0$ as 'there is a particle on the site x with energy level $\eta(x)$'. A particle at site x moves with rate $c_{ex}(\eta(x))$ to a nearest neighbor site y if y is vacant. A unit of energy of the particle at site x is transferred to a nearest neighbor particle at site y with rate $c_{ge}(\eta(x))$, if $\eta(y) < M$.

The dual chain shall act on the state space $\mathscr{Y}$ of $\mathscr{T}_\emptyset^M$ defined by

$$\mathscr{Y} := \big\{(B_1, \ldots, B_M) \in \mathscr{T}_\emptyset^M : B_j \cap B_k = \emptyset,\ j \neq k,\ j,k = 1, \ldots, M\big\}.$$

Define $H : \mathscr{X} \times \mathscr{Y} \to \{0,1\}$ by

$$H(\eta, \mathbf{B}) := \prod_{i=1}^{M} \prod_{x\in B_i} \delta_i(\eta(x)), \quad \eta \in \mathscr{X},\ \mathbf{B} = (B_1, \ldots, B_M) \in \mathscr{Y}.$$

Obviously, for each $\mathbf{B} \in \mathscr{Y}$, it holds that $H(\cdot, \mathbf{B}) \in \mathrm{T}(\mathscr{X}) \subset C(\mathscr{X})$.

One checks easily that

$$H(\eta^{x\to y}, \mathbf{B}) = H(\eta, \mathbf{B}^{y\to x}) \ \text{ and } \ H(\eta^{x,y}, \mathbf{B}) = H(\eta, \mathbf{B}^{x,y}), \quad \eta \in \mathscr{X},\ \mathbf{B} \in \mathscr{Y},$$

where, for $\mathbf{B} \in \mathscr{Y}$,

$$\mathbf{B}^{x\to y} := \big(B_1, \ldots, B_{i-1} \cup \{x\}, B_i \setminus \{x\}, \ldots, B_j \setminus \{y\}, B_{j+1} \cup \{y\}, \ldots, B_M\big)$$

if $x \in B_i$, $y \in B_j$, $2 \leq i \leq M$, $1 \leq j \leq M-1$, and $\mathbf{B}^{x\to y} := B$ otherwise; and

$$\mathbf{B}^{x,y} := \big(B_1, \ldots, B_i \setminus \{x\} \cup \{y\}, B_{i+1}, \ldots, B_M\big)$$

if $x \in B_i$, $y \notin \cup_{i=1}^M B_i$ and $\mathbf{B}^{x,y} := \mathbf{B}$ otherwise. It follows that

$$A_{ge}H(\cdot, \mathbf{B})(\eta) = \mathbb{Q}_{ge}H(\eta, \cdot)(\mathbf{B}), \quad \text{and} \quad A_{ex}H(\cdot, \mathbf{B})(\eta) = \mathbb{Q}_{ex}H(\eta, \cdot)(\mathbf{B}), \tag{4.11}$$

$\eta \in \mathscr{X}$, $\mathbf{B} \in \mathscr{Y}$, if one chooses

$$\mathbb{Q}_{ge}(\mathbf{B}, \mathbf{C}) := \sum_{i=2}^{M} \sum_{x\in B_i} \sum_{j=1}^{M-1} \sum_{y\in B_j} \delta_1(|x-y|)C_{ge}(i)\delta_{\mathbf{B}^{y\to x}}(\mathbf{C})$$

and

$$\mathbb{Q}_{ex}(\mathbf{B},\mathbf{C}) := \sum_{i=1}^{M}\sum_{x\in B_i}\sum_{y\in\cup_{k=1}^{M}(C_k\setminus B_k)} \delta_1(|x-y|)C_{ex}(i)\delta_{\mathbf{B}^{x,y}}(\mathbf{C}),$$

$\mathbf{B},\mathbf{C}\in\mathscr{Y}$, $\mathbf{B}\neq\mathbf{C}$. Let

$$\mathbb{Q}_{ge}(\mathbf{B}) := -\mathbb{Q}_{ge}(\mathbf{B},\mathbf{B}) := \sum_{\mathbf{C}\in\mathscr{Y}} \mathbb{Q}_{ge}(\mathbf{B},\mathbf{C}), \quad \mathbf{B}\in\mathscr{Y},$$

and

$$\mathbb{Q}_{ex}(\mathbf{B}) := -\mathbb{Q}_{ex}(\mathbf{B},\mathbf{B}) := \sum_{\mathbf{C}\in\mathscr{Y}} \mathbb{Q}_{ex}(\mathbf{B},\mathbf{C}), \quad \mathbf{B}\in\mathscr{Y}.$$

With

$$|\mathbf{B}| := \sum_{k=1}^{M}|B_k|, \quad \mathbf{B}=(B_1,\ldots,B_M)\in\mathscr{Y},$$

one finds

$$|\mathbf{B}^{x\to y}| = |\mathbf{B}^{x,y}| = |\mathbf{B}|. \tag{4.12}$$

Therefore

$$\begin{aligned} 0 \le \mathbb{Q}_{ge}(\mathbf{B}) &\le \sum_{i=1}^{M} C_{ge}(i)\sum_{x\in B_i}\sum_{j=1}^{M}\sum_{y\in B_j}\delta_1(|x-y|) \\ &\le 2dM\sum_{i=1}^{M} C_{ge}(i)|B_i| \le 2dMC_0|\mathbf{B}|, \end{aligned} \tag{4.13}$$

where $C_0 := \max\{C_{ge}(i), 1\le i\le M\}$. In a similar way, one obtains

$$0 \le \mathbb{Q}_{ge}(\mathbf{B}) \le 2dC_1|\mathbf{B}|, \tag{4.14}$$

where $C_1 := \max\{C_{ex}(i), 1\le i\le M\}$. Hence $\mathbb{Q}_{ge}$ and $\mathbb{Q}_{ex}$ are rate matrices.

Because of (4.11), $\mathbb{Q}_{ge}$, $\mathbb{Q}_{ex}$ and the sum $\mathbb{Q} := \mathbb{Q}_{ge} + \mathbb{Q}_{ex}$ are H-dual to A_{ge}, A_{ex} and A, respectively. In particular, it holds that

$$AH(\cdot,\mathbf{B})(\eta) = \mathbb{Q}H(\eta,\cdot)(\mathbf{B}), \quad \eta\in\mathscr{X},\ \mathbf{B}\in\mathscr{Y}.$$

Both $\mathbb{Q}_{ge}$ and $\mathbb{Q}_{ex}$ satisfy the condition (Q) with respect to the same function $\varphi:\mathscr{Y}\to\mathbb{R}$. Indeed, let $\mathscr{Y}_n := \{\mathbf{B}\in\mathscr{Y} : |B_i|\le n, 1\le i\le M\}$, $n\in\mathbb{N}$, and define $\varphi(\mathbf{B}) := |\mathbf{B}|$, $\mathbf{B}\in\mathscr{Y}$. Then the sets $\mathscr{Y}_n$ increase to $\mathscr{Y}$ with $n\to\infty$ and (Q0) is satisfied. Condition (Q1) is easily checked using (4.12) while (Q2) follows from (4.13) and (4.14). Thus, by Corollary 3.1, the Markov chain defined by $\mathbb{Q}$ is non-exploding and H-dual to the IPS generated by A.

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