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Gibbsian Characterization for the Reversible Measures of Interacting Particle Systems

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Abstract. For a large class of interacting particle systems (IPS) on the d-dimensional square lattice a criterion for reversibility of measures is derived. It is shown that a reversible measure exists if and only if the local processes which the IPS consists of are reversible w.r.t. the same measure. This result is translated into constraints that are put on the family of conditional probabilities of the reversible measure. For spin processes as well as more complex composite IPS a necessary and sufficient condition for the existence of reversible measures is proven and it is shown that the reversible measures coincide with the Gibbs measures corresponding to a specification that is constructed directly from the transition rates.

KEYWORDS: reversible interacting particle system, Gibbs measure, detailed balance condition, spin process

AMS SUBJECT CLASSIFICATION: 60K35, 82C22

1. Introduction

Interacting particle systems (IPS) model the temporal evolution of spatially extended systems that consist of many locally interacting entities. We consider IPS that are Feller – Markov processes on some configuration space $\mathbb{X} = W^S$, where W is a finite set and $S = \mathbb{Z}^d$ is the d-dimensional square lattice, $d = 1, 2, \ldots$ The dynamics of these IPS is given by a transition mechanism that can be interpreted to be the superposition of many local or microscopic transitions.

We call an invariant measure ν of a given IPS globally reversible, if the IPS started in ν is time-reversal invariant (Definition 2.14). This property follows from the slightly stronger property of local or microscopic reversibility of the measure ν. The latter means that each of the local transition mechanisms is time-reversal invariant with respect to ν (Definition 2.15).

Originally, IPS were introduced to analyze equilibrium problems in statistical physics, in particular to get a better understanding of the phenomenon of phase transitions for Gibbs measures. For that purpose, IPS were studied that are locally reversible with respect to a given family of Gibbs measures [5,6,8,10–12,20]. Since Gibbs measures are random fields which are described via specifications and interaction potentials [9], the transition rates of the so-constructed IPS are naturally related to a specification or a potential given in advance. This relationship between the transition mechanism of the IPS and the potential manifests itself in the detailed balance equations.

It has become clear in the meantime that interacting particle systems are interesting models in their own right. They may be successfully applied when one wants to describe non-equilibrium behavior. In this case, one specifies the transition rates of an IPS according to the underlying concepts of the local dynamics, but often without connection to a potential. For an analysis of the long-time behavior, the set of invariant measures should be described. So it is natural to ask, under which conditions a given IPS has globally reversible measures and whether one can further characterize them. However, if transition rates are given which are not constructed from a potential and if one does not succeed in guessing such a potential, the standard detailed balance equations are of no avail in the search for globally reversible measures. Thus there is a need for a method which applies to those situations where no potential is given in advance. Note that this problem is in some respect inverse to the original approach, since now the transition rates are the starting point and (all) random fields which are globally reversible with respect to this dynamics shall be determined.

We develop a reversibility criterion for a very general class of IPS. We show that under very weak additional assumptions local reversibility is a necessary condition for a given IPS to be globally reversible (Theorem 3.1). The resulting system of local reversibility conditions can be translated into requirements on the family of conditional distributions of a candidate measure, which lead to a system of generalized detailed balance equations (DB) (Corollary 3.1). Thus we can show that the existence of a globally reversible measure is equivalent to the solvability of the system of generalized detailed balance equations in the set of probability measures.

As mentioned above, it is already well-known that the condition of detailed balance (DB) is sufficient for global reversibility. However, the question whether (DB) is also a necessary condition for global reversibility was not yet considered in full generality. Corresponding statements were derived only for spin-flip processes [14, Prop. IV.2.7] and particle jump processes [8, Lemma (2.15)].

The system of generalized detailed balance equations that we derive is formulated in terms of the transition rates only. This allows to decide on the existence of globally reversible measures by looking on the solvability of (DB) in the set of probability measures. This solvability essentially comprises two conditions:

(1) Each local transition mechanism is reversible.

(2) The system of solutions of the individual detailed balance conditions can be composed into a probability measure with according conditional distributions.

Since the local mechanisms are Markov chains with finite state space, it is typically not too hard to check (1). To satisfy condition (2), it is necessary that the system of individual solutions is consistent. In this case however, the system of solutions can be understood as subfamily of a specification. Under fairly natural continuity and positivity assumptions, such a subfamily can be extended to a Gibbsian specification. This extension is unique, if the subfamily is sufficiently large. See [2] for details in the case of one-point families and [3] for more general results. If the transition rates are given, then their continuity and positivity properties are inherited by the conditional distributions of any measure ν which satisfies the detailed balance condition. Thus any solution of (DB) and consequently any globally reversible measure is a Gibbs measure, when the appropriate assumptions on the continuity and the positivity of the transition rates are imposed.

We apply our method to the example spin processes, which are IPS where at an event only the state at one lattice site changes but a finite number of states is allowed at each fixed lattice site, and then to some more complex IPS. Thereby we contribute manageable criteria for the existence of globally reversible measures. Specializing our results to the well-studied class of spin-flip processes, we supplement the results in [14, § IV.2] with a necessary and sufficient condition for the existence of globally reversible measures which involves only the family of transition rates of the given IPS. Note that these findings are in agreement with the results of Mu Fa Chen and coworkers [1], where the global reversibility of spin-flip processes is studied with the help of so-called potentiality.

For translation invariant IPS, Maes, Redig and Verschuere [15, 16] studied the question whether there are locally reversible measures. They introduced the concept of mean entropy production $MEP(\rho, c)$ for a given translation-invariant invariant measure ρ of the IPS with transition rates c. They could show that the mean entropy production $MEP(\rho, c)$ vanishes if and only if ρ and c satisfy the generalized detailed balance condition, which is equivalent to local reversibility. Our work complements their results in several aspects. Firstly, since we have shown that local reversibility is necessary for time-reversal invariance, one can conclude now from their results that $MEP(\rho, c)$ vanishes if and only if ρ is globally reversible. Secondly, the mean entropy production $MEP(\rho, c)$ is an asymptotic quantity which is obtained from a thermodynamic limit procedure, while our approach to analyze under which conditions the system of detailed balance equations is solvable in the set of probability measures leads to local criteria for global reversibility. Thirdly, their methods apply to translation-invariant IPS, while for our findings no translation-invariance is assumed.

2. Preliminaries

2.1. Configuration space

Let $S := \mathbb{Z}^d, d \geq 1$, be the d-dimensional square lattice and take the symbol $\mathcal{T}$ for the set of all non-empty finite subsets of S. For singletons in $\mathcal{T}$ we usually write x instead of $\{x\}$. Further fix some $n \in \mathbb{N}, n \geq 2$, and put $W := \{0, 1, \ldots, n-1\}$ equipped with the discrete metric and the Borel-σ-algebra $\mathcal{W} := \mathcal{B}(W)$ (which coincides with the power set of W). Let λ denote the uniform distribution on $(W, \mathcal{W})$, it shall be our reference measure on W. Take the *configuration space* $\mathbb{X} := W^S$ as the state space of the Markov processes that shall be studied. The space $\mathbb{X}$ endowed with the product topology of the discrete topology of W is compact and metrizable. It shall be equipped with the Borel-σ-algebra $\mathcal{F} := \mathcal{B}(\mathbb{X})$. Let $\mathscr{P}(\mathbb{X}, \mathcal{F})$ denote the set of probability measures on $(\mathbb{X}, \mathcal{F})$.

For each $T \subset S$, the set $S \setminus T$ shall be denoted by T^c, $\mathbb{X}_T = W^T$ will represent the configuration space over T and

$$\pi_T : \mathbb{X} \longrightarrow \mathbb{X}_T : \pi_T(\eta) := (\eta(x))_{x \in T} =: \eta_T$$

denotes the *projection* from $\mathbb{X}$ onto $\mathbb{X}_T$. We write $\mathcal{F}_T$ for the σ-algebra

$$\mathcal{F}_T := \sigma(\pi_T) = \pi_T^{-1}\Big(\bigotimes_{x \in T} \mathcal{W}\Big)$$

and introduce

$$\mathcal{C}_T := \mathcal{F}_{S \setminus T}.$$

Definition 2.1.

(1) Suppose that $\mathcal{E} \subset \mathcal{F}$ is a σ-field. A function $\Psi : \mathcal{F} \to \mathcal{E}$ is a *selection homomorphism* for $\mathcal{E}$, if $\Psi(B) = B$, $B \in \mathcal{E}$.

(2) A family $\mathfrak{E} = (\mathcal{E}_T)_{T \in \mathcal{T}}$ of σ-fields $\mathcal{E}_T \subset \mathcal{F}$ is called *appropriate* if the following conditions are satisfied:

(a) $\mathcal{C}_T \subset \mathcal{E}_T$, $T \in \mathcal{T}$;

(b) $\mathcal{E}_{T_1} \supset \mathcal{E}_{T_2}$ for any T_1, $T_2 \in \mathcal{T}$ with $T_1 \subset T_2$;

(c) there exists a compatible family $(\Psi_T)_{T\in\mathcal{T}}$ of selection homomorphisms for $\mathfrak{E}$, i.e. Ψ_T is a selection homomorphism for $\mathcal{E}_T$, $T \in \mathcal{T}$, and $\Psi_{T_2} \circ \Psi_{T_1} = \Psi_{T_2}$ for $T_1, T_2 \in \mathcal{T}$ with $T_1 \subset T_2$.

Remark 2.1.

(1) The family $(\mathcal{C}_T)_{T\in\mathcal{T}}$ is appropriate.
Indeed, define $\Psi_T := \pi_T^{-1} \circ \pi_T$, where

$$\pi_T(A) = \{\pi_T(\eta) : \eta \in A\}, \quad T \in \mathcal{T}.$$

Then Ψ_T is a selection homomorphism and $\Psi_{T_2} \circ \Psi_{T_1} = \Psi_{T_2}$ for $T_1, T_2 \in \mathcal{T}$ with $T_1 \subset T_2$.

(2) Define

$$N_T^w : \mathbb{X} \longrightarrow \mathbb{R} : N_T^w(\eta) := \sum_{x\in T} \mathbb{1}\{\pi_x^{-1}(w)\}(\eta), \quad \eta \in \mathbb{X},$$

where $\mathbb{1}\{A\}$ is the indicator function of a set $A \in \mathcal{F}$. Then the family $(\mathcal{E}_T)_{T\in\mathcal{T}}$ with $\mathcal{E}_T = \sigma(N_T^w, \pi_V : w \in W, V \subset T^c)$ is appropriate.
Indeed, for $T \in \mathcal{T}$, define

$$\Psi_T(A) := \pi_{T^c}^{-1} \circ \pi_{T^c}(A) \cap \bigcap_{w\in W} (N_T^w)^{-1} \circ N_T^w(A), \quad A \in \mathcal{F}.$$

Then $\Psi(A) \in \mathcal{E}_T$, $A \in \mathcal{F}$, and $\Psi_{T_2} \circ \Psi_{T_1} = \Psi_{T_2}$ for T_1, $T_2 \in \mathcal{T}$ with $T_1 \subset T_2$.

For $u \in \mathbb{X}_T$, $\eta \in \mathbb{X}$, let $\tau_T(\eta, u)$ be the configuration where η_T is replaced by u, that is

$$\tau_T(\eta, u)(z) = \begin{cases} \eta(z), & z \in T^c \\ u(z), & z \in T. \end{cases} \tag{2.1}$$

By $C(\mathbb{X})$ we denote the set of continuous real functions on $\mathbb{X}$ equipped with the sup-norm $\|\cdot\|$. We write $C(\mathbb{X}, \mathcal{A})$ for the linear space of all $\mathcal{A}$-measurable continuous functions, where $\mathcal{A}$ is some sub-σ-field of $\mathcal{F}$, and we use the shorthand $C_T(\mathbb{X})$ for the linear space of all $\mathcal{F}_T$-measurable continuous functions, $T \in \mathcal{T}$. Let $T(\mathbb{X}) = \bigcup_{T\in\mathcal{T}} C_T(\mathbb{X})$ denote the set of all so-called *local* functions. Note that $C(\mathbb{X})$ is the uniform closure of $T(\mathbb{X})$. Given $f \in C(\mathbb{X})$, we define with

$$\mathrm{tm}(f) := \big\{x \in S : \sup\{|f(\eta) - f(\zeta)| : \eta_{x^c} = \zeta_{x^c}\} > 0\big\}$$

the support of f. One easily checks that $f \in T(\mathbb{X}) \Leftrightarrow \mathrm{tm}(f) \in \mathcal{T}$.

For $\xi \in \mathbb{X}, T \in \mathcal{T}$, put

$$\mathbb{1}\{\xi_T\} := \mathbb{1}\{\pi_T^{-1}(\xi_T)\},$$

where $\mathbb{1}\{A\}$ is the indicator function of a set $A \in \mathcal{F}$. Further define $\mathbb{1}\{\eta_\emptyset\} := 1$, $\eta \in \mathbb{X}$. With slight abuse of notation we will use the same symbols for the corresponding indicator functions on $\mathbb{X}_\Delta$, $\Delta \supset T$. Let

$$E_T(\mathbb{X}) := \{\mathbb{1}\{\xi_T\} : \xi \in \mathbb{X}\}, \quad T \in \mathcal{T} \cup \{\emptyset\}$$

and set

$$E(\mathbb{X}) := \bigcup_{T \in \mathcal{T} \cup \{\emptyset\}} E_T(\mathbb{X}),$$

the set of *simple* functions. We observe that $C_T(\mathbb{X})$ is the linear hull of $E_T(\mathbb{X})$, that is $C_T(\mathbb{X}) = \operatorname{span} E_T(\mathbb{X})$, $T \in \mathcal{T}$, and

$$T(\mathbb{X}) = \operatorname{span}(E(\mathbb{X})) = \bigcup_{V \in \mathcal{T}, V \supset T} C_V(\mathbb{X}), \quad T \in \mathcal{T}.$$

Further, given $f \in C(\mathbb{X})$, $\nu \in \mathscr{P}(\mathbb{X})$, we denote

$$\nu(f) := \int f \, d\nu.$$

Definition 2.2.

(1) Given $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ we denote by

$$\operatorname{supp}(\nu) := \{\eta \in \mathbb{X} : \nu(U_\delta(\eta)) > 0 \quad \text{for all} \quad \delta > 0\}$$

the *support* of ν. Here $U_\delta(\eta)$ is the open ball in the metric space $\mathbb{X}$ with center $\eta \in \mathbb{X}$ and radius $\delta > 0$.

(2) $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ is called *dense*, if $\operatorname{supp}(\nu) = \mathbb{X}$.

Proposition 2.1. *Suppose that $\lambda \in \mathscr{P}(W, \mathscr{P}(W))$ is the uniform distribution on W.*

(1) *The measure*

$$\lambda^S := \bigotimes_{x \in S} \lambda$$

is dense.

(2) *A sufficient condition for a measure $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ to be dense is that $\lambda^S \ll \nu$ on $\mathcal{F}_T$ for all $T \in \mathcal{T}$.*

(3) *If* $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ *is dense and* $f \in C(\mathbb{X})$, *then* $f = 0$ (ν-a.s.) *implies that* $f \equiv 0$.

Proof.

(1) Fix $\eta \in \mathbb{X}$, $\delta > 0$. Since $U_\delta(\eta) \supset \pi_T^{-1}(\eta_T)$ for sufficiently large T and $\lambda^S(\pi_T^{-1}(\eta_T)) = \lambda^T(\eta_T) > 0$, we find that $\eta \in \operatorname{supp}(\lambda^S)$.

(2) We find that $\nu(A) > 0$ for any $A \in \cup_{T \in \mathcal{T}} \mathcal{F}_T$ with $A \neq \emptyset$ since $\lambda^S(A) > 0$ for those sets. Hence ν is strictly positive on the cylinder sets. Further for each $\eta \in \mathbb{X}, \delta > 0$ we have $U_\delta(\eta) \supset \pi_T^{-1}(\eta_T)$ for sufficiently large T. Therefore ν is dense.

(3) Suppose that $f(\eta) > a$ for some $\eta \in \mathbb{X}$ and $a > 0$. Then, by continuity of f, there exists a $\delta > 0$ such that $f(\zeta) > a/2 > 0$ for all $\zeta \in U_\delta(\eta)$. But since ν is dense, we conclude that $\nu(U_\delta(\eta)) > 0$ which contradicts the property $f = 0$ (ν-a.s.). □

2.2. Gibbs measures

Definition 2.3. A family $\gamma = (\gamma_T)_{T \in \mathcal{T}}$ of maps $\gamma_T : \mathcal{F} \times \mathbb{X} \to [0, 1]$ is called a *specification*, if

(1) $\gamma_T(\cdot, \eta)$ is a probability measure on $(\mathbb{X}, \mathcal{F})$ for all $\eta \in \mathbb{X}$, $T \in \mathcal{T}$;

(2) $\gamma_T(A, \cdot)$ is $\mathcal{C}_T$-measurable for all $A \in \mathcal{F}$, $T \in \mathcal{T}$;

(3) (γ_T) is *proper*, i.e. $\gamma_T(A \cap B, \cdot) = \gamma_T(A, \cdot)\mathbb{1}\{B\}$, $A \in \mathcal{F}$, $B \in \mathcal{C}_T$, $T \in \mathcal{T}$;

(4) (γ_T) is *consistent*, i.e. $\gamma_T\gamma_V = \gamma_V$ for all $T, V \in \mathcal{T}$, $T \subset V$, where

$$\gamma_T\gamma_V(A, \eta) = \int \gamma_V(A, \zeta)\gamma_T(d\zeta, \eta), \quad \eta \in \mathbb{X},\ A \in \mathcal{F}.$$

Further define the set of all probability measures on $(\mathbb{X}, \mathcal{F})$ which are *specified by* γ via

$$\mathscr{G}(\gamma) := \big\{\nu \in \mathcal{P}(\mathbb{X}, \mathcal{F}) : \nu(A \mid \mathcal{C}_T) = \gamma_T(A, \cdot)\ (\nu\text{-a.s.}),\ A \in \mathcal{F},\ T \in \mathcal{T}\big\}.$$

Given some subfamily $\gamma^0 = (\gamma_T)_{T \in \mathcal{T}_0}$ where $\mathcal{T}_0 \subset \mathcal{T}$, we set

$$\mathscr{G}(\gamma^0) := \big\{\nu \in \mathcal{P}(\mathbb{X}, \mathcal{F}) : \nu(A \mid \mathcal{C}_T) = \gamma_T(A, \cdot)\ (\nu\text{-a.s.}),\ A \in \mathcal{F},\ T \in \mathcal{T}_0\big\}.$$

Remark 2.2. Replacing the σ-fields $\mathcal{C}_T$ in the above definition by the σ-fields $\mathcal{E}_T$ of an appropriate filtration $\mathfrak{E} = (\mathcal{E}_T)_{T \in \mathcal{T}}$ of $\mathcal{F}$, the concept of a specification is generalized to that of an $\mathfrak{E}$-specification $\gamma^{\mathfrak{E}}$. If, for instance,

$$\mathcal{E}_T = \sigma(\pi_{T^c}, N_T^w : w \in W), \quad T \in \mathcal{T},$$

then the measures which are specified by $\gamma^{\mathfrak{E}}$ are canonical (Gibbs) measures.

Given a specification $\gamma = (\gamma_T)$, a measurable function f and a set $T \in \mathcal{T}$, let us denote by $\gamma_T f$ the measurable function that is given by

$$\gamma_T f(\eta) := \int f(\zeta)\,\gamma(d\zeta,\eta), \quad \eta \in \mathbb{X}.$$

Similarly, define for $\nu \in \mathscr{P}(\mathbb{X},\mathcal{F})$, $T \in \mathcal{T}$, the probability measure $\nu\gamma_T$ by

$$\nu\gamma_T(A) = \int \gamma_T(A,\eta)\,\nu(d\eta), \quad A \in \mathcal{F}.$$

Definition 2.4.

(1) A specification $\gamma = (\gamma_T)$ is called *positive*, if

$$\gamma_T(\pi_T^{-1}(v),\eta) > 0, \quad v \in \mathbb{X}_T,\ \eta \in \mathbb{X},\ T \in \mathcal{T}.$$

(2) A specification $\gamma = (\gamma_T)$ is called *continuous* or *quasi-local*, if, for each $T \in \mathcal{T}$, $f \in C(\mathbb{X})$ implies that $\gamma_T f \in C(\mathbb{X})$.

Proposition 2.2.

(1) *If some specification γ is positive then, for each $T \in \mathcal{T}$, any two measures ν, $\mu \in \mathscr{G}(\gamma)$ are mutually absolutely continuous on $\mathcal{F}_T$. In addition, $\nu \in \mathscr{G}(\gamma)$ and λ^S are mutually absolutely continuous on $\mathcal{F}_T$ for each $T \in \mathcal{T}$.*

(2) *If some specification γ is positive, then any $\nu \in \mathscr{G}(\gamma)$ is dense.*

Proof.
(1) [9, (1.28)(2)]

(2) It follows from (1) that $\lambda^S \ll \nu$ on $\mathcal{F}_T$ for any $T \in \mathcal{T}$. Together with Proposition 2.1 (2) the assertion is proven. □

Definition 2.5.

(1) A family $\Phi = (\Phi_A)_{A\in\mathcal{T}}$ of functions $\Phi_A : \mathbb{X} \to \mathbb{R}$ is a *potential*, if both of the following properties are satisfied:

(a) For each $A \in \mathcal{T}$, the function Φ_A is $\mathcal{F}_A$-measurable.

(b) For any $\Lambda \in \mathcal{T}$, $\eta \in \mathbb{X}$, the net

$$H_\Lambda^\Phi(\eta) := \sum_{\substack{A\in\mathcal{T},\\ A\cap\Lambda\neq\emptyset}} \Phi_A(\eta) \tag{2.2}$$

converges.

(2) A potential is called *uniformly convergent*, if for all $\Lambda \in \mathcal{T}$ the convergence in (2.2) is uniform with respect to $\eta \in \mathbb{X}$.

(3) A potential is called *absolutely summable*, if $\sum_{A\in\mathcal{T},A\ni x}\sup_{\eta\in\mathbb{X}}|\Phi_A(\eta)|$ is finite for all $x \in S$.

(4) A potential is said to be of *finite range* if for all $x \in S$ there exists a set $\Delta_x \in \mathcal{T}$ such that $\Phi_A \equiv 0$ for any $A \ni x$ unless $A \subset \Delta_x$.

Definition 2.6. A specification is said to be *Gibbsian*, if there is a uniformly convergent potential $\Phi = (\Phi_A)_{A\in\mathcal{T}}$ such that $\gamma = \gamma^\Phi$, where

$$\gamma_T^\Phi(\pi_T^{-1}(u),\eta) := \big(Z_T^\Phi(\eta)\big)^{-1}\exp\big\{-H_T^\Phi(\tau_T(\eta,u))\big\},\quad u\in\mathbb{X}_T,\ \eta\in\mathbb{X},\ T\in\mathcal{T},$$

and

$$Z_T^\Phi(\eta) := \sum_{u\in\mathbb{X}_T}\exp\big\{-H_T^\Phi(\tau_T(\eta,u))\big\},\quad \eta\in\mathbb{X},\ T\in\mathcal{T}.$$

The normalizing factor Z_T^Φ is called the *partition function.*

Any measure $\nu \in \mathscr{G}(\gamma)$ that is specified by the Gibbsian specification γ is called *Gibbs measure* (w.r.t. γ).

Specifications describe random fields on $\mathbb{Z}^d$. But considering the consistency condition in Definition 2.3 (4), one finds that they are 'too rich', i.e. they contain redundant information. In the following we address the question whether already a family $(\gamma_x)_{x\in S}$ specifies a random field on $\mathbb{Z}^d$. We refer to the work of Dachian and Nahapetian [2] where most results that are relevant here can be found. Note that there are several more recent papers such as [3] and [7], where generalizations of the stated criteria were derived.

Proposition 2.3. *Suppose that $\gamma = (\gamma_T)_{T\in\mathcal{T}}$ is a specification.*

(1) *If γ satisfies $\gamma_x(A,\eta) > 0$ for each $x \in S$, $\eta \in \mathbb{X}$ and each atom A of $\mathcal{F}_x$, then γ is positive and any $\nu \in \mathscr{G}(\gamma)$ is dense.*

(2) *If $\gamma_x(\pi_x^{-1}(v),\cdot)$ is positive and continuous for any $x \in S$, $v \in W$, then γ is positive and continuous. In addition, it holds that*

$$\emptyset \neq \mathscr{G}(\gamma) = \big\{\nu \in \mathscr{P}(\mathbb{X},\mathcal{F}) : \nu = \nu\gamma_x,\ x\in S\big\}.$$

Proof.

(1) We show that $\gamma_{\{x,y\}}(A,\eta) > 0$ for each $x, y \in S$ with $x \neq y$, any $\eta \in \mathbb{X}$ and each atom A of $\mathcal{F}_{\{x,y\}}$. Then it follows that γ is positive by induction over the size of T and, by Prop. 2.2 (2), any $\nu \in \mathscr{G}(\gamma)$ is dense.

Since γ is proper and consistent and since $\gamma_{\{x,y\}}(\cdot,\cdot)$ is $\mathcal{C}_{\{x,y\}}$-measurable in the second argument, we have for any $u, v \in W$,

$$\underbrace{\gamma_{\{x,y\}}\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v), \cdot\big)}_{\text{does not depend on the coordinate at } y}$$
$$= \gamma_{\{x,y\}}\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v), \tau_y(\cdot, v)\big) = \gamma_{\{x,y\}}\gamma_x\big(\pi_x^{-1}(u) \cap \underbrace{\pi_y^{-1}(v)}_{\in \mathcal{C}_x}, \tau_y(\cdot, v)\big)$$
$$= \gamma_{\{x,y\}}\big(\mathbb{1}\{\pi_y^{-1}(v)\} \underbrace{\gamma_x\big(\pi_x^{-1}(u), \tau_y(\cdot, v)\big)}_{\mathcal{C}_{\{x,y\}}\text{-measurable}}\big)$$
$$= \gamma_x\big(\pi_x^{-1}(u), \tau_y(\cdot, v)\big)\gamma_{\{x,y\}}\big(\mathbb{1}\{\pi_y^{-1}(v)\}\big)$$
$$= \gamma_x\big(\pi_x^{-1}(u), \tau_y(\cdot, v)\big)\gamma_{\{x,y\}}\big(\pi_y^{-1}(v), \cdot\big)$$
$$= \gamma_x\big(\pi_x^{-1}(u), \tau_y(\cdot, v)\big)\gamma_{\{x,y\}}\big(\gamma_y(\pi_y^{-1}(v), \cdot)\big).$$

Since $\gamma_x\big(\pi_x^{-1}(u), \tau_y(\cdot, v)\big)$ and $\gamma_y\big(\pi_y^{-1}(v), \cdot\big)$ are positive by assumption, it follows that

$$\gamma_{\{x,y\}}\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v), \cdot\big) > 0.$$

(2) Define

$$p_x(\eta) := \gamma_x\big(\pi_x^{-1}(\eta(x)), \eta\big)\lambda(\eta(x)), \quad x \in S,\ \eta \in \mathbb{X}.$$

Then p_x is continuous and thus, since $\mathbb{X}$ is compact, bounded and uniformly positive. Now let $T \in \mathcal{T}$ be the union of two disjoint sets $T_1, T_2 \in \mathcal{T}$ and suppose that continuous functions p_{T_1}, p_{T_2} have already been constructed which are continuous as well as bounded and uniformly positive. We define

$$p_T = \frac{p_{T_1}}{\lambda_{T_1}(p_{T_1}/p_{T_2})}.$$

Since (λ_T) is an independent specification, we find that $\lambda_{T_1}(p_{T_1}/p_{T_2})$ is continuous since p_{T_1}/p_{T_2} is continuous. Hence p_T is continuous as well as uniformly finite and positive. As was proven in [9, Theorem (1.33)], the functions which are constructed this way satisfy $\gamma_T = p_T\lambda_T$, $T \in \mathcal{T}$, thus γ is continuous.

The equality $\mathcal{G}(\gamma) = \{\nu \in \mathcal{P}(\mathbb{X}, \mathcal{F}) : \nu = \nu\gamma_x, x \in S\}$ was also derived in [9, Theorem (1.33)]. What is left to show is that $\mathcal{G}(\gamma) \neq \emptyset$. But this follows from [9, Theorem (4.22)], since λ is finite and all p_T's are bounded. □

Definition 2.7. A specification $\gamma = (\gamma_T)_{T\in\mathcal{T}}$ is said to be a *vacuum specification* (with vacuum 0) or *weakly positive* if

$$\gamma_T\big(\pi_T^{-1}(\mathbf{0}_T), \zeta\big) > 0, \quad T \in \mathcal{T},\ \zeta \in \mathbb{X},$$

where $\mathbf{0} \in \mathbb{X}$ is defined by $\mathbf{0}(z) = 0$, $z \in S$.

Note that any positive specification is a vacuum specification.

Definition 2.8.

(1) A family $p = (p_x)_{x\in S}$ of $(\mathbb{X}, \mathcal{C}_x)$-$(W, \mathcal{W})$-probability kernels is said to be *consistent*, if for any $x, y \in S$ and $\zeta \in \mathbb{X}$ with $\zeta(x) = \zeta(y) = 0$ the following property is satisfied:

$$\begin{aligned} &p_x(\zeta, v)p_y(\tau_x(\zeta, v), u)p_x(\tau_y(\zeta, u), 0)p_y(\zeta, 0) \\ &\qquad = p_y(\zeta, u)p_x(\tau_y(\zeta, u), v)p_y(\tau_x(\zeta, v), 0)p_x(\zeta, 0). \end{aligned}$$

(2) A family $\gamma^0 = (\gamma_x)_{x\in S}$ of proper $(\mathbb{X}, \mathcal{C}_x)$-$(\mathbb{X}, \mathcal{F})$-probability kernels is said to be *consistent*, if the related family $p = (p_x)_{x\in S}$ defined via

$$p_x(\zeta, u) := \gamma_x(\pi_x^{-1}(u), \zeta), \quad x \in S, \zeta \in \mathbb{X}, u \in W,$$

is consistent.

Definition 2.9. A family $\gamma^0 = (\gamma_x)_{x\in S}$ of $(\mathbb{X}, \mathcal{C}_x)$-$(\mathbb{X}, \mathcal{F})$-probability kernels is said to be a *one-point specification*, if there exists a vacuum specification $\tilde{\gamma} = (\tilde{\gamma}_T)_{T\in\mathcal{T}}$ such that γ^0 is a subfamily of $\tilde{\gamma}$, i.e. $\gamma_x = \tilde{\gamma}_x$, $x \in S$.

Proposition 2.4. *A family $\gamma^0 = (\gamma_x)_{x\in S}$ of proper $(\mathbb{X}, \mathcal{C}_x)$-$(\mathbb{X}, \mathcal{F})$-probability kernels is a one-point specification if and only if it is consistent.*

Proof. Apply [2, Thm. 3.4] to the family

$$h = \{h_x^\zeta(v) : x \in S,\ \zeta \in \mathbb{X} \text{ with } \zeta(x) = \zeta(y) = 0, v \in W\},$$

where

$$h_x^\zeta(v) := \frac{p_x(\zeta, v)}{p_x(\zeta, 0)}, \quad x \in S,\ \zeta \in \mathbb{X},\ v \in W.$$

□

Proposition 2.5. *Suppose that $\gamma^0 = (\gamma_x)_{x\in S}$ is a one-point specification.*

(1) *If γ^0 is continuous, that is if $f \in C(\mathbb{X})$ implies $\gamma_x f \in C(\mathbb{X})$ for any $x \in S$, then*

$$\mathscr{G}(\gamma^0) = \{\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F}) : \nu\gamma_x = \nu,\ x \in S\} \neq \emptyset.$$

(2) *If γ^0 is positive and continuous, then there is a unique continuous positive specification which contains γ^0 as a subfamily. Further, the equation $\mathscr{G}(\gamma^0) = \mathscr{G}(\gamma)$ is satisfied and there exists a uniformly convergent (vacuum) potential, such that γ is Gibbsian with respect to this potential.*

Proof.
(1) When γ^0 is a continuous one-point specification, there exists a vacuum specification γ which contains γ^0 as a subfamily and is continuous as well, see [2, Thm. 4.1] and its straightforward generalization to arbitrary finite W. Following [4], we find $\mathscr{G}(\gamma) \neq \emptyset$. Since $\mathscr{G}(\gamma^0) \supset \mathscr{G}(\gamma)$, the assertion is proven.

(2) Since γ^0 is a one-point specification, there exists a vacuum specification γ which contains γ^0 as a subfamily. Using the fact that γ^0 is positive, it follows from [9, Thm. (1.33)] that γ is positive, uniquely determined by γ^0 and $\mathscr{G}(\gamma) = \mathscr{G}(\gamma^0)$. The existence of a uniformly convergent vacuum potential Φ such that $\gamma = \gamma^\Phi$ is deduced directly from [9, Cor. (2.31)]. □

Proposition 2.6. *For each* $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$, *there exists a specification* $\gamma^\nu = (\gamma_T^\nu)_{T\in\mathcal{T}}$ *such that*

$$\nu(A \mid \mathcal{C}_T) = \gamma_T^\nu(A, \cdot) \quad \nu\text{-a.s.}, \ A \in \mathcal{F}.$$

Proof. We apply [18, Thm. 3.3]: S is a countable set and $\mathcal{W}$ is finite, hence $\mathcal{F} = \bigotimes_{x\in S} \mathcal{W}$ is countably generated and standard Borel. Thus any probability measure $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ is *perfect* in the sense of [18]. Since $(\mathcal{C}_T)_{T\in\mathcal{T}}$ is an appropriate filtration of $\mathcal{F}$, there exists a compatible family of selection homomorphisms. □

Remark 2.3. The statement of the above proposition remains true if $(\mathcal{C}_T)_{T\in\mathcal{T}}$ is replaced by an appropriate filtration $\mathfrak{E} = (\mathcal{E}_T)_{T\in\mathcal{T}}$ of $\mathcal{F}$. The corresponding $\mathfrak{E}$-specification is denoted by $\gamma^{\mathfrak{E},\nu} = (\gamma_T^{\mathfrak{E},\nu})_{T\in\mathcal{T}}$.

2.3. Interacting particle systems

Interacting particle systems (IPS) are Markov processes with state space $\mathbb{X}$. They are described by defining the dynamics within finite sub-volumes of S (local dynamics) via families of transition rates. More precisely, for the current configuration $\eta \in \mathbb{X}$ and fixed $T \in \mathcal{T}$, a transition involving the coordinates in T is described by $\tau_T(\eta, u)$, where $u \in \mathbb{X}_T$ is the new (local) configuration that is observed within T after the transition has taken place (cp. (2.1)). The rate at which this transition occurs shall be specified by some transition rate $c_T(\eta, u)$.

Definition 2.10. Suppose that we are given a family $c = (c_T)_{T\in\mathcal{T}}$ of non-negative functions $c_T : \mathbb{X} \times \mathbb{X}_T \to [0, \infty)$. If, for each $u \in \mathbb{X}_T$, the function $c_T(\cdot, u)$ is continuous, then c is a *family of transition rate functions.*

Sometimes it is more convenient to use the following notation,

$$c_T(\eta, u, v) := c_T(\tau_T(\eta, u), v), \quad u, v \in \mathbb{X}_T, \ T \in \mathcal{T}, \ \xi \in \mathbb{X}.$$

We define

$$c_T(x) := \sup\left\{|c_T(\eta, u) - c_T(\zeta, v)| : \eta_{\{x\}^c} = \zeta_{\{x\}^c}\right\}, \quad x \in S,\ T \in \mathcal{T},$$

and

$$c_T := \sup_{\eta \in \mathbb{X}} \sum_{u \in \mathbb{X}_T} c_T(\eta, u), \quad T \in \mathcal{T}.$$

Definition 2.11. Denote

$$\mathcal{T}_0 := \{T \in \mathcal{T} : \sup_{\eta \in \mathbb{X}} c_T(\eta, \mathbb{X}_T) > 0\}$$

and suppose that $\mathcal{T}_0 \neq \emptyset$. A family $c = (c_T)_{T \in \mathcal{T}}$ of transition rate functions is

(1) *admissible*, if

$$\sup_{x \in S} \sum_{T \ni x} c_T < \infty, \tag{2.3}$$

and

$$\sup_{x \in S} \sum_{T \ni x} \sum_{z \neq x} c_T(z) < \infty; \tag{2.4}$$

(2) *finite*, if for any $x \in S$ there is a set $T_x \in \mathcal{T}$ such that $V \subset T_x$ for any $V \in \mathcal{T}_0$ with $V \ni x$;

(3) *local*, if $c_T(\cdot, v) \in T(\mathbb{X})$, $\quad T \in \mathcal{T}_0, v \in \mathbb{X}_T$;

(4) *translation invariant*, if

$$c_T(\eta, u) = c_{\theta_y T}(\theta_y \eta, u),$$

where $\theta_y T := \{x + y : x \in T\}$ and $\theta_y \eta(x) = \eta(x - y)$, $x \in S$ and $y \in S$, $\eta \in \mathbb{X}$, $T \in \mathcal{T}$;

(5) *of finite range*, if it is translation invariant and if there is a $R \in \mathbb{R}$ such that, for any $T \in \mathcal{T}_0$,

$$|x - y| \leq R, \quad x, y \in T \cup \bigcup_{u \in \mathbb{X}_T} \mathrm{tm}(c_T(\cdot, u)).$$

Remark 2.4. We have that (2), (3), (4) $\Leftrightarrow$ (5) $\Rightarrow$ (1).

Definition 2.12.

(1) A family of admissible rate functions is *standard*, if, for any $T \in \mathcal{T}_0$, $\eta \in \mathbb{X}$, it holds that $c_T(\eta, v) = 0$ for any $v \in \mathbb{X}_T$ with $v(x) = \eta(x)$ for some $x \in T$.

(2) A family of admissible rate functions is *positive*, if, for any $T \in \mathcal{T}_0$, $\eta \in \mathbb{X}$, it holds that $c_T(\eta, v) > 0$ for any $v \in \mathbb{X}_T$ with $v(x) \neq \eta(x)$ for some $x \in T$.

(3) A family of admissible rate functions has got the *two-way communication property* (*twc*), if, for any $T \in \mathcal{T}_0$, $\eta \in \mathbb{X}$, it holds that $c_T(\eta, u, v) > 0 \Leftrightarrow c_T(\eta, v, u) > 0$.

(4) A family of admissible rate functions is *irreducible*, if, for any $T \in \mathcal{T}_0$, $\eta \in \mathbb{X}$, it holds that $(c_T(\eta, u, v))_{u,v \in \mathbb{X}_T}$ is an irreducible transition matrix on $\mathbb{X}_T$.

Suppose we are given a family $c = (c_T(\cdot, \cdot))_{T \in \mathcal{T}}$ of admissible transition rates. We define an operator $A : T(\mathbb{X}) \to C(\mathbb{X})$ by

$$Af(\eta) = \sum_{T \in \mathcal{T}} \sum_{v \in \mathbb{X}_T} c_T(\eta, v)\big[f(\tau_T(\eta, v)) - f(\eta)\big], \quad \eta \in \mathbb{X}, \ f \in T(\mathbb{X}).$$

By [14, Prop. I.3.2], A is well-defined if c is admissible. Further, according to [14, Thm. I.3.9], the closure of A is a Markov generator which generates a Markov semigroup $(T_t)_{t \geq 0}$ on $C(\mathbb{X})$. The corresponding Markov process with cadlag-trajectories is called *interacting particle system* (*IPS*).

Note that A can be interpreted as the superposition of *local operators* A_T: $C(\mathbb{X}) \to C(\mathbb{X})$, where

$$A_T f(\eta) = \sum_{v \in \mathbb{X}_T} c_T(\eta, v)\big[f(\tau_T(\eta, v)) - f(\eta)\big], \quad \eta \in \mathbb{X}, \ f \in C(\mathbb{X}),$$

i.e.

$$Af = \sum_{T \in \mathcal{T}} A_T f, \quad f \in T(\mathbb{X}).$$

It is easily checked that A_T is bounded for each $T \in \mathcal{T}$. Further we find that

$$\sum_{T \in \mathcal{T}} \|A_T f\| \leq C\|f\|, \quad f \in T(\mathbb{X}), \tag{2.5}$$

for some constant $C > 0$, cp. the proof of [14, Prop. I.3.2].

For fixed $T \in \mathcal{T}$, the operator A_T can be regarded to be the Markov generator of an IPS which corresponds to the family of transition rates $\tilde{c} = (\tilde{c}_V)_{V \in \mathcal{T}}$ where

$$\tilde{c}_V(\eta, u) = \begin{cases} c_T(\eta, u), & V = T, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta \in \mathbb{X}, \ u \in \mathbb{X}_T, \ V \in \mathcal{T}.$$

The family $\tilde{c}$ is admissible when c is admissible, therefore the operator A_T is a Markov operator. Given an initial configuration η, the IPS that is generated by A_T only changes the configuration within the finite sub-lattice T while the configuration that is seen in T^c remains unchanged. Therefore A_T can be interpreted to specify the local dynamics within T given a so-called external condition η_{T^c}.

Definition 2.13. For $T \in \mathcal{T}_0$, the Markov generator A_T is called *conservative* w.r.t. some σ-field $\mathcal{E} \subset \mathcal{F}$ if

$$A_T(fg) = f A_T g, \quad f \in C(\mathbb{X}, \mathcal{E}),\ g \in T(\mathbb{X}).$$

Remark 2.5.

(1) It is easily checked that each A_T is $\mathcal{C}_\Lambda$-conservative, $\Lambda \supset T$, $T \in \mathcal{T}$.

(2) Considering *exclusion processes* (*with speed change*), where $c_T(\eta, u) = 0$, $T \in \mathcal{T}$, $\eta \in \mathbb{X}$, $u \in \mathbb{X}_T$, unless $T = \{x, y\}$ with $x, y \in S$ and $\eta(x) = u(y)$ and $\eta(y) = u(x)$, we find that, for $x, y \in S$, the operator $A_{\{x,y\}}$ is $\sigma(\pi_{V^c}, N_V)$-conservative for each V with $x, y \in V$. Here $N_V = (N_V^0, \ldots, N_V^n) : \mathbb{X} \to \mathbb{N}^n$ is the particle number within volume $V \in \mathcal{T}$ as defined in Remark 2.1.

Remark 2.6.

(1) A family (c_T) of admissible and finite transition rates can be transformed into an *equivalent* family of admissible and standard transition rates $(\tilde{c}_T)$ by a standardization procedure. Define for $T \in \mathcal{T}$, $v \in \mathbb{X}_T$, $\eta \in \mathbb{X}$,

$$\tilde{c}_T(\eta, v) := \begin{cases} 0, & \text{if } v(x) = \eta(x) \text{ for some } x \in T, \\ \displaystyle\sum_{\Delta \in \mathcal{T}_0, \Delta \supset T} c_\Delta(\eta, (\eta_{\Delta \setminus T}, v)), & \text{otherwise,} \end{cases}$$

where $(\eta_{\Delta\setminus T}, v) \in \mathbb{X}_\Delta$ is the composition of v and $\eta_{\Delta\setminus T}$, i.e.

$$(\eta_{\Delta\setminus T}, v)(z) = \eta(z), \quad z \in \Delta \setminus T,$$

and

$$(\eta_{\Delta\setminus T}, v)(z) = v(z), \quad z \in T.$$

Since (c_T) is finite, there exits, for any $x \in S$, a set $T_x \in \mathcal{T}$ such that $V \subset T_x$ for any $V \in \mathcal{T}_0$ with $V \ni x$. Hence, for any $T \in \mathcal{T}$, it follows from $\Delta \in \mathcal{T}$, $\Delta \supset T$ that

$$\Delta \subset \bigcup_{x \in T} T_x \in \mathcal{T}.$$

So the above sum defining $\tilde{c}_T$ is a finite sum. The family $(\tilde{c}_T)$ is equivalent to (c_T) in the sense that the specified IPS' coincide.

Indeed, define $\Delta_T(u, v) := \{x : u(x) \neq v(x)\}$, $u, v \in \mathbb{X}_T$ and consider

$$\tilde{A} f(\eta) = \sum_{T \in \mathcal{T}} \sum_{v \in \mathbb{X}_T} \tilde{c}_T(\eta, v) \left[f(\tau_T(\eta, v)) - f(\eta) \right]$$

for $\eta \in \mathbb{X}, f \in C(\mathbb{X})$. Then we find

$$\begin{aligned}
\tilde{A}f(\eta) &= \sum_{T \in \mathcal{T}} \sum_{\substack{v \in \mathbb{X}_T: \\ \Delta_T(v,\eta)=T}} \sum_{\substack{\Lambda \in \mathcal{T}: \\ \Lambda \supset T}} c_\Lambda(\eta, (\eta_{\Lambda \setminus T}, v))\big[f(\tau_T(\eta, v)) - f(\eta)\big] \\
&= \sum_{T \in \mathcal{T}} \sum_{\substack{v \in \mathbb{X}_T: \\ \Delta_T(v,\eta)=T}} \sum_{\substack{\Lambda \in \mathcal{T}: \\ \Lambda \supset T}} c_\Lambda(\eta, (\eta_{\Lambda T}, v))\big[f(\tau_\Lambda(\eta, (\eta_{\Lambda T}, v))) - f(\eta)\big] \\
&= \sum_{\Lambda \in \mathcal{T}} \sum_{T \subset \Lambda} \sum_{\substack{u \in \mathbb{X}_\Lambda: \\ \Delta_\Lambda(u,\eta)=T}} c_\Lambda(\eta, u)\big[f(\tau_\Lambda(\eta, u)) - f(\eta)\big] \\
&= \sum_{\Lambda \in \mathcal{T}} \sum_{u \in \mathbb{X}_T} c_\Lambda(\eta, u)\big[f(\tau_\Lambda(\eta, u)) - f(\eta)\big] \\
&= Af(\eta), \qquad \eta \in \mathbb{X},\ f \in C(\mathbb{X}).
\end{aligned}$$

(2) If the transition rates are so that $\mathcal{T}_0 = \{\{x\}, x \in S\}$ (so-called *spin processes*), then the rates are always finite and standard.

(3) So-called *edge processes*, which are defined by the requirements that $W = \{0,1\}$, $\mathcal{T}_0 = \{\{x,y\},\ x,y \in S,\ x \neq y\}$ and $c_T(\eta, u) = q(\eta_T, u)$, $T \in \mathcal{T}_0$, for some non-negative matrix $q : W^2 \times W^2 \to [0,\infty)$, are in general not standard. Only in the special case that $q(u,v) = 0$ unless $u = (0,1)$, $v = (1,0)$ or $u = (1,0)$, $v = (0,1)$ (simple exclusion process), the corresponding edge process is standard.

Definition 2.14. Let A denote the Markov generator corresponding to a family c of admissible transition rates. A measure $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ is called *globally reversible* with respect to A if

$$\int gAf\, d\nu = \int fAg\, d\nu, \quad f,g \in T(\mathbb{X}). \tag{GR}$$

The set of all globally reversible probability measures w.r.t. A (resp. c) is denoted by $\mathscr{R}(c)$.

Remark 2.7. This concept of global reversibility is equivalent to time-reversal invariance of the IPS with generator A and initial measure ν. See for instance [14, § II.5] for details.

Definition 2.15. Let $A = \sum_{T \in \mathcal{T}} A_T$ denote the Markov generator corresponding to a family $c = (c_T)_{T \in \mathcal{T}}$ of admissible transition rates. A measure $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$ is called *locally reversible* with respect to A if

$$\int gA_T f\, d\nu = \int fA_T g\, d\nu, \quad f,g \in T(\mathbb{X}),\ T \in \mathcal{T}. \tag{LR}$$

3. Main results

Theorem 3.1. *Suppose that a family $c = (c_T)_{T\in\mathcal{T}}$ of admissible transition rates is finite and standard. Then ν is globally reversible if and only if it is locally reversible, that is $\nu \in \mathscr{R}(c)$ if and only if*

$$\int gA_T f\,d\nu = \int fA_T g\,d\nu, \quad T \in \mathcal{T},\ f,g \in C(\mathbb{X}).$$

Remark 3.1. In the case that $W = \{0,1\}$, $\mathcal{T}_0 = \{\{x\} : x \in S\}$, Liggett [14, Ch. IV] (Stochastic Ising models) infers from global reversibility upon local invariance, i.e. it is concluded that $\nu \in \mathscr{R}(c)$ implies $\int A_x f\,d\nu = 0$, $f \in C(\mathbb{X})$ for any $x \in S$. Similarly, for exclusion processes with speed change where $\mathcal{T}_0 = \{\{x,y\} : x,y \in S\}$, Georgii [8, § 2] proved that global reversibility implies local invariance in the sense that $\int A_{\{x,y\}} f\,d\nu = 0$, $f \in C(\mathbb{X})$ for any $x,y \in S$. In the first case, the state space W of the local mechanism has only two elements. In the latter case, the closed classes of the local transition mechanism, which are of the form $\{(u,v),(v,u)\}$ with $u,v \in W$, always contain at most two elements, although the single-site space can have more than two elements. Therefore, in both cases, local invariance and local reversibility coincide and the mentioned results are specializations of the above theorem.

Before proving the theorem we state the following lemma.

Lemma 3.1. *Suppose that $\eta,\zeta \in \mathbb{X}$, $V \in \mathcal{T}$ such that $\Delta := \{x \in V : \eta(x) \neq \zeta(x)\} \neq \emptyset$ and set $f := \mathbb{1}\{\eta_V\}$, $g := \mathbb{1}\{\zeta_V\}$. Then the following properties hold:*

(1) $(fA_\Lambda g - gA_\Lambda f) = 0$, $\Lambda \in \mathcal{T}$, *unless* $\Lambda \cap V = \Delta$.

(2) $$(fAg - gAf) = \sum_{\Lambda\in\mathcal{T}_0,\Lambda\cap V=\Delta} (fA_\Lambda g - gA_\Lambda f).$$

Proof. Let us first verify the following propositions.

(a) $f(\xi)g(\xi) = 0$, $\xi \in \mathbb{X}$,

(b) $f(\xi)g(\tau_\Lambda(\xi,v)) = 0$ for any $\xi \in \mathbb{X}$, $v \in \mathbb{X}_\Lambda$, with $v(x) \neq \xi(x)$, $x \in \Lambda$, $\Lambda \in \mathcal{T}$, unless $\Lambda \cap V = \Delta$,

(c) $f(\tau_\Lambda(\xi,v))g(\xi) = 0$ for any $\xi \in \mathbb{X}$, $v \in \mathbb{X}_\Lambda$, with $v(x) \neq \xi(x)$, $x \in \Lambda$, $\Lambda \in \mathcal{T}$, unless $\Lambda \cap V = \Delta$.

Property (a) is immediately clear. For the proof of (b), we fix $\xi \in \mathbb{X}$, $\Lambda \in \mathcal{T}$, $v \in \mathbb{X}_\Lambda$ with $v(x) \neq \xi(x)$, $x \in \Lambda$, and use the shorthand $\xi^v := \tau_\Lambda(\xi,v)$. We estimate

$$\begin{aligned} 0 \le f(\xi)g(\xi^v) &= \mathbb{1}\{\eta_V\}(\xi)\,\mathbb{1}\{\zeta_V\}(\xi^v) \\ &\le \mathbb{1}\{\eta_{(\Lambda\cap V)\setminus\Delta}\}(\xi)\,\mathbb{1}\{\zeta_{(\Lambda\cap V)\setminus\Delta}\}(\xi^v)\,\mathbb{1}\{\eta_{\Delta\setminus\Lambda}\}(\xi)\,\mathbb{1}\{\zeta_{\Delta\setminus\Lambda}\}(\xi^v) \\ &= \mathbb{1}\{\eta_{(\Lambda\cap V)\setminus\Delta}\}(\xi)\,\mathbb{1}\{\zeta_{(\Lambda\cap V)\setminus\Delta}\}(v)\,\mathbb{1}\{\eta_{\Delta\setminus\Lambda}\}(\xi)\,\mathbb{1}\{\zeta_{\Delta\setminus\Lambda}\}(\xi). \end{aligned} \tag{3.1}$$

The latter equality follows from the fact that $\xi^v_\Lambda = v$ and $\xi^v_{\Lambda^c} = \xi_{\Lambda^c}$. We observe that

$$\mathbb{1}\{\eta_{(\Lambda\cap V)\setminus\Delta}\}(\xi)\,\mathbb{1}\{\zeta_{(\Lambda\cap V)\setminus\Delta}\}(v) = 1 \iff \begin{cases} (\Lambda\cap V)\setminus\Delta = \emptyset \\ \text{or} \\ (\Lambda\cap V)\setminus\Delta \neq \emptyset \ \text{ and } \ \eta_{(\Lambda\cap V)\setminus\Delta} = \zeta_{(\Lambda\cap V)\setminus\Delta} = v_{(\Lambda\cap V)\setminus\Delta}, \end{cases}$$

and

$$\mathbb{1}\{\eta_{\Lambda\setminus\Delta}\}(\xi)\,\mathbb{1}\{\zeta_{\Lambda\setminus\Delta}\}(\xi) = 1 \iff \begin{cases} \Delta\setminus\Lambda = \emptyset \\ \text{or} \\ \Delta\setminus\Lambda \neq \emptyset \ \text{ and } \ \eta_{\Delta\setminus\Lambda} = \zeta_{\Delta\setminus\Lambda} = \xi_{\Delta\setminus\Lambda}. \end{cases}$$

Since $v(x) \neq \xi_\Lambda(x)$, $x \in \Lambda$, and $\eta_\Delta(x) \neq \zeta_\Delta(x)$, $x \in \Delta$, we conclude that the product in (3.1) vanishes for $\Lambda \cap V \neq \Delta$, which proves (b).

The validity of proposition (c) follows analogously.

Now we get for $\Lambda \in \mathcal{T}$, $\xi \in \mathbb{X}$,

$$\begin{aligned}(f\,A_\Lambda g - gA_\Lambda f)(\xi) &\\ = \sum_{v\in\mathbb{X}_\Lambda} c_\Lambda(\xi,v)\big(f(\xi)g(\xi^v) - f(\xi)g(\xi) - g(\xi)f(\xi^v) + g(\xi)f(\xi)\big) &\\ = \sum_{\substack{v\in\mathbb{X}_\Lambda:\\ v(x)\neq\xi(x),x\in\Lambda}} c_\Lambda(\xi,v)\big(f(\xi)g(\xi^v) - g(\xi)f(\xi^v)\big) = 0 \quad \text{unless } \ \Lambda\cap V = \Delta &\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(fAg - gAf) &= \sum_{\Lambda\in\mathcal{T}} (fA_\Lambda g - gA_\Lambda f) \\ &= \sum_{\Lambda\in\mathcal{T}_0,\Lambda\cap V=\Delta} (fA_\Lambda g - gA_\Lambda f).\end{aligned} \tag{3.3}$$

□

Proof of Theorem 3.1. Suppose that $f, g \in T(\mathbb{X})$. For any sequence $(T_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ with $T_n \subset T_{n+1}$ and $\cup_{n\in\mathbb{N}} T_n = S$, we get from (2.5) the estimate

$$\begin{aligned}\Big\|\sum_{T\subset T_n} (gA_T f - fA_T g)\Big\| &\leq \sum_{T\in\mathcal{T}} \|gA_T f - fA_T g\| \\ &\leq \|g\| \sum_{T\subset\mathcal{T}} \|A_T f\| + \|f\| \sum_{T\subset\mathcal{T}} \|Ag\| \\ &\leq 2C\|f\|\|g\|.\end{aligned}$$

Therefore the Dominated Convergence Theorem implies that

$$\begin{aligned}\sum_{T\in\mathcal{T}}\int (gA_Tf - fA_Tg)\,d\nu &= \int \sum_{T\in\mathcal{T}} (gA_Tf - fA_Tg)\,d\nu \\ &= \int \Big(g\sum_{T\in\mathcal{T}} A_Tf - f\sum_{T\in\mathcal{T}} A_Tg\Big)\,d\nu \\ &= \int (gAf - fAg)\,d\nu.\end{aligned}$$

Thus it follows from (LR) by summation over $T\in\mathcal{T}$ that (GR) is satisfied, i.e. $\nu\in\mathscr{R}(c)$.

We show now that (GR) implies (LR). To show this define

$$\mathcal{T}^0 := \{T\in\mathcal{T}_0 : \Lambda\in\mathcal{T}_0, \Lambda\supset T \Longrightarrow \Lambda = T\},$$

$$\mathcal{T}^n := \Big\{T\in\mathcal{T}_0\setminus\bigcup_{j=0}^{n-1}\mathcal{T}^j : \Lambda\in\mathcal{T}_0\setminus\bigcup_{j=0}^{n-1}\mathcal{T}^j, \Lambda\supset T \Longrightarrow \Lambda = T\Big\}, \quad n\in\mathbb{N}.$$

Since (c_T) is finite, $\mathcal{T}^0\neq\emptyset$ and

$$\mathcal{T}_0 = \bigcup_{j\in\mathbb{N}_0}\mathcal{T}^j.$$

Fix $T\in\mathcal{T}^0$. We show that (LR) is satisfied for $f,g\in E_V(\mathbb{X})$ with $V\supset T$. To this end, fix $\eta,\zeta\in\mathbb{X}$, $V\in\mathcal{T}$, $V\supset T$, and set

$$\Delta := \{x\in V : \eta(x)\neq\zeta(x)\}, \quad f := \mathbb{1}\{\eta_V\}, \quad g := \mathbb{1}\{\zeta_V\}.$$

Without loss of generality we can assume that $\Delta\neq\emptyset$ since (LR) is trivially satisfied for $f=g$. It follows from Lemma 3.1 (1) that

$$\int (fA_Tg - gA_Tf)\,d\nu = 0, \quad T\neq\Delta,$$

and by Lemma 3.1 (2) we have

$$\begin{aligned}\int (fA_Tg - gA_Tf)\,d\nu &= \int (fAg - gAf)\,d\nu \\ &\quad - \sum_{\Lambda\in\mathcal{T}_0,\Lambda\neq T,\Lambda\cap V = T}\int (fA_\Lambda g - gA_\Lambda f)\,d\nu, \quad T=\Delta.\end{aligned}$$

Since $T\in\mathcal{T}^0$ implies that $\Lambda\notin\mathcal{T}_0$ for any $\Lambda\supsetneq T$, it follows that

$$\sum_{\Lambda\in\mathcal{T}_0,\,\Lambda\neq T,\,\Lambda\cap V=T}\int (fA_\Lambda g - gA_\Lambda f)\,d\nu = 0.$$

Hence $\nu \in \mathscr{R}(c)$ implies that

$$\int (fA_T g - gA_T f)\, d\nu = 0. \tag{3.4}$$

Since $C_V(\mathbb{X}) = \mathrm{span} E_V(\mathbb{X})$ and $T(\mathbb{X}) = \bigcup_{V \in \mathcal{T}, V \supset T} C_V(\mathbb{X})$, the last equation extends to $f, g \in T(\mathbb{X})$.

Now assume that (3.4) is valid for $T \in \bigcup_{j=0}^{n-1} \mathcal{T}^j$ and $f, g \in T(\mathbb{X})$. Fix $T \in \mathcal{T}^n$ and choose $f, g \in E_V(\mathbb{X})$ with $V \supset T$. Suppose that

$$f = \mathbb{1}\{\eta_V\}, \quad g = \mathbb{1}\{\zeta_V\} \quad \text{for some } \eta, \zeta \in \mathbb{X}$$

with

$$\Delta := \{x \in V : \eta(x) \neq \zeta(x)\} \neq \emptyset.$$

We observe that the relation $\Lambda \in \mathcal{T}_0$, $\Lambda \supsetneq T$ implies that $\Lambda \in \bigcup_{j=0}^{n-1} \mathcal{T}^j$. Hence

$$\sum_{\Lambda \in \mathcal{T}_0, \Lambda \neq T, \Lambda \cap V = T} \int (fA_\Lambda g - gA_\Lambda f)\, d\nu = 0,$$

by our assumption of induction. Applying Lemma 3.1 we get from $\nu \in \mathscr{R}(c)$ that

$$\int (fA_T g - gA_T f)\, d\nu$$
$$= \begin{cases} 0, & \text{if } T \neq \Delta, \\ \int (fAg - gAf)\, d\nu - \sum_{\substack{\Lambda \in \mathcal{T}_0, \Lambda \neq T, \\ \Lambda \cap V = T}} \int (fA_\Lambda g - gA_\Lambda f)\, d\nu = 0, & \text{if } T = \Delta. \end{cases}$$

Again, since $C_V(\mathbb{X}) = \mathrm{span} E_V(\mathbb{X})$ and $T(\mathbb{X}) = \bigcup_{V \in \mathcal{T}, V \supset T} C_V(\mathbb{X})$, the last equation extends to $f, g \in T(\mathbb{X})$. By induction we conclude that

$$\int (fA_T g - gA_T f)\, d\nu = 0, \quad T \in \mathcal{T}, \ f, g \in T(\mathbb{X}).$$

The linear operator A_T on $C(\mathbb{X})$ is bounded and thus continuous. Since the set $T(\mathbb{X})$ is dense in $C(\mathbb{X})$, the last equation is valid for any $f, g \in C(\mathbb{X})$. □

Proposition 3.1. *Suppose that $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$, $T \in \mathcal{T}_0$. Let $\mathcal{F}_T \subset \mathcal{F}$ be a σ-field such that A_T is conservative with respect to $\mathcal{E}_T$ with $\mathcal{C}_T \subset \mathcal{E}_T$. Then the following statements are equivalent.*

(1) $\int (gA_T f - fA_T g)\, d\nu = 0$, $f, g \in C(\mathbb{X})$.

(2) $\nu(gA_T f - fA_T g \mid \mathcal{E}_T) = 0$, $f, g \in T(\mathbb{X})$, ν-a.s.

(3) $\gamma_T^\nu(gA_Tf - fA_Tg) = 0,\ \ f, g \in T(\mathbb{X})$, *$\nu$-a.s.*

(4) *For ν-a.a. $\xi \in \mathbb{X}$ it holds that*

$$c_T(\xi, u, v)\gamma_T^\nu(\pi_T^{-1}(u), \xi) = c_T(\xi, v, u)\gamma_T^\nu(\pi_T^{-1}(v), \xi), \tag{DB}$$

for all $u, v \in \mathbb{X}_T$, where $c(\eta, u, v) := c(\tau_T(\eta, u), v)$, $u, v \in \mathbb{X}_T$, $\eta \in \mathbb{X}$.

Remark 3.2.

(1) We will use the phrase that *a specification $\gamma = (\gamma_T)_{T\in\mathcal{T}}$ resp. some subfamily $\gamma^0 = (\gamma_T)_{T\in\mathcal{T}_0}$ solves* (DB), if (DB) is satisfied for *each* $\xi \in \mathbb{X}$ and $T \in \mathcal{T}_0$.

(2) For any fixed $T \in \mathcal{T}_0$, $\xi \in \mathbb{X}$ (actually only the *external condition* ξ_{T^c} counts), statement (DB) is the property that the Markov chain on $\mathbb{X}_T$ with generator matrix $(c_T(\xi, u, v))_{u,v\in\mathbb{X}_T}$ is reversible with respect to the measure $p_T(\xi, \cdot) \in \mathscr{P}(\mathbb{X}_T, \mathcal{W}^T)$ given by $p_T(\xi, u) := p_T(\xi, \{u\}) := \gamma_T^\nu(\pi_T^{-1}(u), \xi)$. Therefore Kolmogorov's criterion [17] on the existence of a reversible measure for finite Markov chains is an essential tool for the decision whether there is a reversible measure for a given IPS. If the local dynamics $(c_T(\xi, u, v))_{u,v\in\mathbb{X}_T}$ are irreducible and reversible, the values of $\gamma_T^\nu(\cdot, \xi)$ are positive and uniquely determined by $(c_T(\xi, u, v))_{u,v\in\mathbb{X}_T}$.

Proof of Proposition 3.1. Fix $T \in \mathcal{T}$. We choose $\eta, \zeta \in \mathbb{X}$, $V \in \mathcal{T}$ with $V \supset T$. Since A_T is $\mathcal{E}_T$ conservative, we conclude from (1) that, for any $h \in C(\mathbb{X}, \mathcal{E}_T)$ and $f = \mathbb{1}\{\eta_V\}h$, $g = \mathbb{1}\{\zeta_V\}$,

$$\begin{aligned}
0 &= \int (gA_Tf - fA_Tg)\, d\nu \\
&= \int \nu\big((gA_Tf - fA_Tg) \mid \mathcal{E}_T\big)\, d\nu \\
&= \int \nu\big((\mathbb{1}\{\zeta_V\}A_T(\mathbb{1}\{\eta_V\}h) - \mathbb{1}\{\eta_V\}hA_T\mathbb{1}\{\zeta_V\}) \mid \mathcal{E}_T\big)\, d\nu \\
&= \int \nu\big(h(\mathbb{1}\{\zeta_V\}A_T\mathbb{1}\{\eta_V\} - \mathbb{1}\{\eta_V\}A_T\mathbb{1}\{\zeta_V\}) \mid \mathcal{E}_T\big)\, d\nu \\
&= \int h\nu\big((\mathbb{1}\{\zeta_V\}A_T\mathbb{1}\{\eta_V\} - \mathbb{1}\{\eta_V\}A_T\mathbb{1}\{\zeta_V\}) \mid \mathcal{E}_T\big)\, d\nu.
\end{aligned}$$

Hence

$$\nu(gA_Tf - fA_Tg \mid \mathcal{E}_T) = 0 \quad \nu\text{-a.s.}, \quad T \in \mathcal{T},\ f, g \in E_V(\mathbb{X}).$$

This equation extends to $f, g \in C(\mathbb{X}, \mathcal{F}_V)$ by the linearity of integration and since $T(\mathbb{X}) = \bigcup_{V\in\mathcal{T}} C(\mathbb{X}, \mathcal{F}_V)$, this yields (2).

Since A_T is always $\mathcal{C}_T$ conservative, we conclude from (2) that

$$0 = \nu(gA_Tf - fA_Tg \mid \mathcal{C}_T) = \gamma_T^\nu(gA_Tf - fA_Tg), \quad \nu\text{-a.s.}, \; T \in \mathcal{T}, \; f, g \in T(\mathbb{X}),$$

which is (3).

Further, we may integrate (3) with respect to ν and get from the property $\nu\gamma_T^\nu = \nu$ that $\int (gA_Tf - fA_Tg)\, d\nu = 0$, $f, g \in T(\mathbb{X})$. The linear operator A_T on $C(\mathbb{X})$ is bounded and continuous, the set $T(\mathbb{X})$ is dense in $C(\mathbb{X})$, thus the last equation is valid for any $f, g \in C(\mathbb{X})$, which is statement (1).

For the proof of the equivalence (3) $\Leftrightarrow$ (DB), we decompose any function $f \in C_V(\mathbb{X})$ with $V \supsetneq T$ into

$$f = \sum_{v \in \mathbb{X}_{V\setminus T}} \mathbb{1}\{\pi_{V\setminus T}^{-1}(v)\} f_v,$$

with $f_v(\eta) := f(\tau_{V\setminus T}(\eta, v))$. Since $f_v \in C_T(\mathbb{X})$, $\mathbb{1}\{\pi_{V\setminus T}^{-1}(v)\} \in C(\mathbb{X}, \mathcal{C}_T)$, $v \in \mathbb{X}_{V\setminus T}$, we find that (3) is equivalent to

$$\gamma_T^\nu(gA_Tf - fA_Tg) = 0, \quad f, g \in C_T(\mathbb{X}), \quad \nu\text{-a.s.}$$

Now $f \in C_T(\mathbb{X})$ is decomposed into a finite sum of simple functions,

$$f = \sum_{u \in \mathbb{X}_T} \mathbb{1}\{\pi_T^{-1}(u)\} c_u,$$

where $c_u = f(\tau_T(\eta, u)) \in \mathbb{R}$. Thus (3) is equivalent to

$$\gamma_T^\nu(gA_Tf - fA_Tg) = 0, \quad f, g \in E_T(\mathbb{X}), \quad \nu\text{-a.s.}$$

Since for $\eta, \zeta \in \mathbb{X}$,

$$\big(\mathbb{1}\{\eta_T\} A_T \mathbb{1}\{\zeta_T\}\big)(\rho) = \mathbb{1}\{\eta_T\}(\rho) c_T(\rho, \zeta_T), \quad \rho \in \mathbb{X},$$

the last equation is equivalent to

$$\int \big(\mathbb{1}\{\eta_T\}(\rho) c_T(\rho, \zeta_T) - \mathbb{1}\{\zeta_T\}(\rho) c_T(\rho, \eta_T)\big)\, \gamma_T^\nu(d\rho, \cdot) = 0, \quad \nu\text{-a.s.}$$

Replacing $\eta_T = u, \zeta_T = v$, this is just (DB). □

Corollary 3.1. *Suppose that the family c of transition rates is admissible, standard and finite. Further assume that $\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F})$. Then the following statements hold true.*

(1) *If $\nu \in \mathscr{R}(c)$ then ν-a.s.*

$$\nu\big(\pi_T^{-1}(u) \mid \mathcal{C}_T\big) c_T(\cdot, u, v) = \nu\big(\pi_T^{-1}(v) \mid \mathcal{C}_T\big) c_T(\cdot, v, u) \quad u, v \in \mathbb{X}_T, \; T \in \mathcal{T}.$$

(2) *If there exists a specification* γ *which solves (DB) for all* $\xi \in \mathbb{X}$, $T \in \mathcal{T}_0$, *then* $\mathscr{G}(\gamma) \subset \mathscr{R}(c)$. *If* c *is additionally irreducible, then* $\mathscr{R}(c) = \mathscr{G}(\gamma)$.

Remark 3.3. In Corollary 3.1 (2) the requirement that there is a specification γ which solves (DB) for all $\xi \in \mathbb{X}$, $T \in \mathcal{T}_0$, actually contains two conditions. Firstly, (DB) has to be solvable for any $\xi \in \mathbb{X}$, $T \in \mathcal{T}_0$. Secondly, one has to decide whether a given family $p = (p_T)_{T \in \mathcal{T}_0}$ of solutions of (DB) allows a *consistent* extension to a specification and whether this extension is unique. In the case that $\mathcal{T}_0 = \{\{x\}, x \in S\}$ the latter question is answered with the help of one-point specifications.

Proof.
(1) The assertion directly follows from Theorem 3.1 and Proposition 3.1.

(2) Suppose that γ is a specification which solves (DB) for all $\xi \in \mathbb{X}$. Then for each $\nu \in \mathscr{G}(\gamma)$ equation (DB) is satisfied ν-a.s. Therefore, by Proposition 3.1 and Theorem 3.1, $\nu \in \mathscr{R}(c)$. Hence $G(\gamma) \subset R(c)$. If c is irreducible, then γ is the unique solution of (DB). Applying (1) we deduce from $\nu \in R(c)$ that

$$\nu\big(\pi_T^{-1}(u) \mid \mathcal{C}_T\big) = \gamma_T\big(\pi_T^{-1}(u), \cdot\big), \quad u \in \mathbb{X}_T,\ T \in \mathcal{T}_0, \quad \nu\text{-a.s.},$$

thus $\nu \in \mathscr{G}(\gamma)$. □

4. Application

4.1. Spin processes

We are going to apply our results to IPS where at most one coordinate is changed in any transition. As before, we have $W = \{0, 1, \ldots, n-1\}$ for some $n \in \mathbb{N}$ and denote by λ the uniform distribution on $(W, \mathcal{W})$.

Definition 4.1. An IPS with transition rates $c = (c_T(\cdot,\cdot))_{T \in \mathcal{T}}$ is a *spin process*, if $\mathcal{T}_0 = \{\{x\} : x \in S\}$. A spin process is a *spin-flip process*, if $W = \{0, 1\}$.

In the following, let us abbreviate $c_x(\eta, v) := c_{\{x\}}(\eta, v)$, $\eta \in \mathbb{X}$, $v \in W$ and $c := (c_x(\cdot,\cdot))_{x \in S}$. Whenever there is no room for misunderstanding, we use the notation A_x, π_x, $\mathcal{C}_x$, x^c etc. instead of $A_{\{x\}}$, $\pi_{\{x\}}$, $\mathcal{C}_{\{x\}}$, $\{x\}^c$ etc. Let us further agree to write

$$c_x(\eta, u, v) := c_x(\tau_x(\eta, u), v), \quad x \in S,\ u, v \in W,\ \eta \in \mathbb{X}.$$

Note that $c_x(\cdot, u, v)$ is $\mathcal{C}_x$-measurable, $u, v \in W$, $x \in S$.

For spin processes, we find that any admissible family of transition rates is finite and standard. Further, by Remark 2.5, for $x \in S$, the Markov operator A_x is $\mathcal{C}_T$-conservative, where $T \in \mathcal{T}$, $T \ni x$.

Proposition 4.1. *Suppose that c is a family of irreducible spin transition rates. Then any globally reversible measure $\nu \in \mathscr{R}(c)$ is dense.*

Proof. Fix $x \in S$. Since c is irreducible, the unique invariant measure $p_x(\xi, \cdot)$ of the transition matrix $c_x(\xi, u, v)_{u,v \in W}$ is positive for any $\xi \in \mathbb{X}$. By Corollary 3.1, it follows from $\nu \in \mathscr{R}(c)$ that $\nu(\cdot \mid \mathcal{C}_x)$ satisfies (DB) ν-a.e. Hence $\nu(\cdot \mid \mathcal{C}_x) = p_x(\xi, \cdot)$ ν-a.e., which implies

$$\nu\big(\pi_x^{-1}(u) \mid \mathcal{C}_x\big) > 0 \quad \nu\text{-a.s..}$$

Hence

$$\nu\big(\pi_x^{-1}(u)\big) = \nu\big(\nu(\pi_x^{-1}(u) \mid \mathcal{C}_x)\big) > 0.$$

Further we have

$$\nu\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v) \mid \mathcal{C}_{\{x,y\}}\big) > 0 \quad \nu\text{-a.s.,}$$

since

$$\begin{aligned}
&\nu\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v) \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\mathbb{1}\{\pi_x^{-1}(u)\}\,\mathbb{1}\{\pi_y^{-1}(v)\} \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\nu\big(\mathbb{1}\{\pi_x^{-1}(u)\}\,\mathbb{1}\{\pi_y^{-1}(v)\} \mid \mathcal{C}_x\big) \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\nu\big(\mathbb{1}\{\pi_x^{-1}(u)\} \mid \mathcal{C}_x\big)\mathbb{1}\{\pi_y^{-1}(v)\} \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\nu\big(\mathbb{1}\{\pi_x^{-1}(u)\} \mid \mathcal{C}_x\big)(\tau_y(\cdot, v))\,\mathbb{1}\{\pi_y^{-1}(v)\} \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\mathbb{1}\{\pi_x^{-1}(u)\} \mid \mathcal{C}_x\big)(\tau_y(\cdot, v))\nu\big(\mathbb{1}\{\pi_y^{-1}(v)\} \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\mathbb{1}\{\pi_x^{-1}(u)\} \mid \mathcal{C}_x\big)(\tau_y(\cdot, v))\nu\big(\nu\big(\mathbb{1}\{\pi_y^{-1}(v)\} \mid \mathcal{C}_y\big) \mid \mathcal{C}_{\{x,y\}}\big)\\
&\quad = \nu\big(\pi_x^{-1}(u) \mid \mathcal{C}_x\big)(\tau_y(\cdot, v))\nu\big(\nu\big(\pi_y^{-1}(v) \mid \mathcal{C}_y\big) \mid \mathcal{C}_{\{x,y\}}\big).
\end{aligned}$$

The latter factor is ν-a.s. positive, and for the former factor we find that

$$\begin{aligned}
&\nu\big(\{\xi : \nu(\pi_x^{-1}(u) \mid \mathcal{C}_x)(\tau_y(\xi, v)) = 0\}\big)\\
&\quad \le \nu\big(\{\xi : \xi(y) = v, \nu(\pi_x^{-1}(u) \mid \mathcal{C}_x)(\xi) = 0\}\big) + \nu\big((\pi_y^{-1}(v))^c\big)\\
&\quad \le \nu\big(\{\xi : \nu(\pi_x^{-1}(u) \mid \mathcal{C}_x)(\xi) = 0\}\big) + \nu\big((\pi_y^{-1}(v))^c\big) = \nu\big((\pi_y^{-1}(v))^c\big) < 1.
\end{aligned}$$

Hence

$$\nu\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v)\big) = \nu\big(\nu\big(\pi_x^{-1}(u) \cap \pi_y^{-1}(v) \mid \mathcal{C}_{\{x,y\}}\big)\big) > 0.$$

It follows by induction that ν is positive on all non-void cylinder sets, hence it is dense. □

Proposition 4.2 (local Kolmogorov criterion). *Suppose we are given a spin process with irreducible, admissible rates $c = (c_x(\cdot, \cdot))_{x \in S}$. Then, for fixed $x \in S$, $\eta \in \mathbb{X}$, the following statements are equivalent.*

(1) *(DB) is solvable for* η, x.

(2) *The transition matrix* $(c_x(\eta, u, v))_{u,v \in W}$ *on* W *has got a reversible measure.*

(3) *The transition matrix* $(c_x(\eta, u, v))_{u,v \in W}$ *on* W *satisfies one of the following equivalent conditions.*

(a) *For each* $k \geq 3$, $v_1, \ldots, v_k \in W$ *with* $v_k = v_1$, *we have*

$$\prod_{i=1}^{k-1} c_x(\eta, v_i, v_{i+1}) = \prod_{i=1}^{k-1} c_x(\eta, v_{i+1}, v_i).$$

(b) $c_x(\eta, \cdot, \cdot)$ *has got the two-way communication property and for each sequence of local states* $v_1, \ldots, v_k$ *with* $k \geq 2$ *and* $c_x(\eta, v_i, v_{i+1}) > 0$, $i = 1, \ldots, k-1$, *the product*

$$\prod_{i=1}^{k-1} \frac{c_x(\eta, v_i, v_{i+1})}{c_x(\eta, v_{i+1}, v_i)}$$

(in general a function of η_{x^c}, $v_1, \ldots, v_k$ *and* k*) is only a function of* η_{x^c}, v_1 *and* v_k.

If any of the preceding statements holds, the solution $p_x(\eta, \cdot)$ *of (DB) is unique in* $\mathscr{P}(W, \mathcal{W})$ *and is given by*

$$p_x(\eta, v) = z_x^{-1}(\eta) \prod_{i=1}^{k-1} \frac{c_x(\eta, v_i, v_{i+1})}{c_x(\eta, v_{i+1}, v_i)}, \quad v \in W \setminus \{0\}, \tag{4.1}$$
$$p_x(\eta, 0) = z_x^{-1}(\eta),$$

where $z_x^{-1}(\eta)$ *is the normalizing factor,* $k \geq 2$ *and* $0 = v_1, \ldots, v_k = v$ *satisfy* $c_x(\eta, v_i, v_{i+1}) > 0$, $i = 1, \ldots, k-1$.

If the rates c *are even positive, the solution* $p_x(\eta, \cdot)$ *of (DB) is given by*

$$p_x(\eta, v) = \left(1 + \sum_{u \in W \setminus \{v\}} \frac{c_x(\eta, v, u)}{c_x(\eta, u, v)}\right)^{-1}, \quad v \in W. \tag{4.2}$$

Proof. Fix $\xi \in \mathbb{X}$, $x \in S$. The transition matrix $(c_x(\eta, u, v))_{u,v \in W}$ defines a finite Markov chain on W which has got a reversible measure iff (DB) is solvable. Hence (1) $\Leftrightarrow$ (2).

The equivalence (2) $\Leftrightarrow$ (3)(b) is a consequence of Kolmogorov's criterion for reversibility (see e.g. [17, Thm. 3.1]) applied to the Q-matrices $(c_x(\xi, u, v))_{u,v \in W}$. There it is also proven that (3)(a) $\Leftrightarrow$ (3)(b) if $c_x(\eta, \cdot, \cdot)$ has got the two-way communication property. So we have to show that the two-way communication property follows from the fact that $c_x(\eta,\cdot,\cdot)$ is irreducible and satisfies (3)(a). Indeed, suppose that there are $u, v \in W$ such that $c_x(\eta, u, v) = 0$ but $c_x(\eta, v, u) > 0$. By irreducibility, there exist $k \geq 3$, $v_1 = u, v_2, \ldots, v_k = v$ such that $c(v_i, v_{i+1}) > 0$. Setting $v_{k+1} := u$, we observe that $\prod_{i=1}^{k} c_x(\eta, v_i, v_{i+1}) > 0$ and $\prod_{i=1}^{k} c_x(\eta, v_{i+1}, v_i) = 0$ because $c_x(\eta, v_{k+1}, v_k) = c_x(\eta, u, v) = 0$. But this contradicts property (3)(a). Thus (2) $\Leftrightarrow$ (3).

If any of the statements (1), (2) or (3) holds, the corresponding solution $p_x(\xi,\cdot)$ of (DB) is unique in the set $\mathscr{P}(W, \mathcal{W})$, since c is irreducible, and it satisfies (4.1). If c is even positive, then we find that

$$p_x(\xi, u) = \frac{c_x(\xi, v, u)}{c_x(\xi, u, v)} p_x(\xi, v), \quad u, v \in W.$$

Hence

$$1 = \sum_{u \in W} p_x(\xi, u) = p_x(\xi, v)\left(1 + \sum_{u \in W\setminus\{v\}} \frac{c_x(\xi, v, u)}{c_x(\xi, u, v)}\right), \quad v \in W,$$

which implies that

$$p_x(\xi, v) = \left(1 + \sum_{u \in W\setminus\{v\}} \frac{c_x(\xi, v, u)}{c_x(\xi, u, v)}\right)^{-1}, \quad v \in W.$$

□

Theorem 4.1. *Suppose we are given a spin process with admissible rates $c = (c_x(\cdot,\cdot))$. If the rates c are irreducible, then the following statements are equivalent:*

(1) $\mathscr{R}(c) \neq \emptyset$.

(2) *(DB) is solvable for all $\xi \in \mathbb{X}$, $x \in S$, the family of solutions being denoted by $p = (p_x)_{x\in S}$, and one of the following equivalent conditions is satisfied.*

(2.a) *For any $x, y \in S$ and $\zeta \in \mathbb{X}$ with $\zeta(x) = \zeta(y) = 0$ it holds that*

$$\begin{aligned} &p_x(\zeta, v)\, p_y(\tau_x(\zeta, v), u) p_x(\tau_y(\zeta, u), 0) p_y(\zeta, 0) \\ &\qquad = p_y(\zeta, u) p_x(\tau_y(\zeta, u), v) p_y(\tau_x(\zeta, v), 0) p_x(\zeta, 0). \end{aligned}$$

(2.b) *For any $x, y \in S$ and $\zeta \in \mathbb{X}$ with $\zeta(x) = \zeta(y) = 0$ it holds that*

$$\begin{aligned} &c_x(\zeta, v)\, c_y(\tau_x(\zeta, v), u) c_x(\tau_{xy}(\zeta, vu), 0) c_y(\tau_y(\zeta, u), 0) \\ &\qquad = c_y(\zeta, u) c_x(\tau_y(\zeta, u), v) c_y(\tau_{xy}(\zeta, vu), 0) c_x(\tau_x(\zeta, v), 0), \end{aligned}$$

where

$$\tau_{xy}(\zeta, vu)(z) = \begin{cases} \zeta(z), & z \neq x,\ z \neq y, \\ v, & z = x, \\ u, & z = y. \end{cases}$$

If any of the preceding statements holds, there is a positive and continuous specification $\gamma = (\gamma_T)_{T\in\mathcal{T}}$ *such that*

$$\mathscr{R}(c) = \mathscr{G}(\gamma) = \{\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F}) : \nu\gamma_x = \nu\}.$$

This specification γ *is uniquely determined by the following equations.*

$$\gamma_x\big(\pi_x^{-1}(v), \eta\big) = z_x^{-1}(\eta) \prod_{i=1}^{k-1} \frac{c_x(\eta, v_i, v_{i+1})}{c_x(\eta, v_{i+1}, v_i)}, \quad v \in W \setminus \{0\}, \quad \gamma_x\big(\pi_x^{-1}(0)\big) = z_x^{-1}(\eta),$$

where $z_x^{-1}(\eta)$ *is the normalizing factor,* $k \geq 2$ *and* $0 = v_1, \ldots, v_k = v$ *with* $c_x(\eta, v_i, v_{i+1}) > 0$, $i = 1, \ldots, k-1$.

Proof. We start with a preliminary consideration. Fix $x \in S$ and suppose that (DB) is solvable for all $\xi \in \mathbb{X}$, the solution being denoted by $p_x(\xi, \cdot)$. Since c is irreducible, the solution is unique in $\mathscr{P}(W, \mathcal{W})$ and positive. We call a finite sequence $(0 = v_1, v_2, \ldots, v_k = v) \in W^k$ a *ζ-path to* $v \in W \setminus \{0\}$, if $k \geq 2$ and $c_x(\zeta, v_i, v_{i+1}) > 0$, $i = 1, \ldots, k-1$. The functions $c_x(\cdot, u, v)$, $u, v \in W$, are continuous, since c is admissible. Hence, for fixed $\xi \in \mathbb{X}$ and fixed ξ-paths to $v \in W \setminus \{0\}$, there exists a $\delta > 0$ such that the ξ-path to v is as well a ζ-path to v, $v \in W \setminus \{0\}$, $\zeta \in U_\delta(\xi)$. This implies that, for any $\zeta \in U_\delta(\xi)$, the representation

$$p_x(\zeta, v) = z_x^{-1}(\zeta) \prod_{i=1}^{k-1} \frac{c_x(\zeta, v_i, v_{i+1})}{c_x(\zeta, v_{i+1}, v_i)}, \quad v \in W \setminus \{0\}, \quad p_x(\zeta, 0) = z_x^{-1}(\zeta),$$

holds true, where $z_x^{-1}(\zeta)$ is the normalizing factor, and $(0 = v_1, \ldots, v_k = v)$, $v \in W \setminus \{0\}$, are the fixed ξ-paths to v. From the continuity of $c_x(\cdot, u, v)$, $u, v \in W$, we conclude that the functions $p_x(\cdot, v)$, $v \in W$, are continuous in ξ. Hence, if (DB) is everywhere solvable, the solution p_x is unique in $\mathscr{P}(W, \mathcal{W})$, and $p_x(\cdot, v)$, $x \in S$, $v \in W$, are positive and continuous functions.

Now we show that (2) implies (1). Suppose that (DB) is solvable for all $\xi \in \mathbb{X}$, $x \in S$, the family of solutions being denoted by $p = (p_x)_{x\in S}$. According to Definition 2.8 p is consistent whenever condition (2.a) is satisfied. Define a family $\gamma^0 = (\gamma_x)_{x\in S}$ of functions $\gamma_x : \mathcal{F}_x \times \mathbb{X} \to [0, 1]$ via

$$\gamma_x\big(\pi_x^{-1}(v), \eta\big) = p_x(\eta, v), \quad v \in W, \ \eta \in \mathbb{X}.$$

Each of these $(\mathbb{X}, \mathcal{C}_x)$-$(\mathbb{X}, \mathcal{F}_x)$ probability kernels has got a unique extension to a proper $(\mathbb{X}, \mathcal{C}_x)$-$(\mathbb{X}, \mathcal{F})$ probability kernel, which shall be denoted by the same symbol. The family γ^0 is consistent since p is consistent and thus it is a positive and continuous one-point specification by Proposition 2.4.

By Proposition 2.5, there exists a specification γ which contains γ^0 as a subfamily and satisfies $\mathscr{G}(\gamma^0) = \mathscr{G}(\gamma) \neq \emptyset$. Clearly, this specification γ is a solution of (DB) and therefore Corollary 3.1 (2) applies. Sine c is irreducible, it follows that $\mathscr{R}(c) = \mathscr{G}(\gamma) = \mathscr{G}(\gamma^0) \neq 0$ which is (1).

It remains to show that (1) implies (2). Suppose that there exists some $\nu \in \mathscr{R}(c)$. Then γ_x^ν satisfies (DB) ν-a.s. Therefore, for any $x \in S$ and ν-a.a. $\xi \in \mathbb{X}$, equation (DB) is solvable. By Proposition 4.2, we find that for each $k \geq 3$, $v_1, \ldots, v_k \in W$ with $v_k = v_1$ and ν-a.a. $\xi \in \mathbb{X}$, the following equality is valid.

$$\prod_{i=1}^{k-1} c_x(\eta, v_i, v_{i+1}) = \prod_{i=1}^{k-1} c_x(\eta, v_{i+1}, v_i). \tag{4.3}$$

But since the functions $c_x(\cdot, v_i, v_{i+1})$ are continuous, $i = 1, \ldots, k$ and since ν is dense, equation (4.3) extends to all $\eta \in \mathbb{X}$. Applying Proposition 4.2 again, it follows that (DB) is solvable for any $\eta \in \mathbb{X}$. The solution is unique in $\mathscr{P}(W, \mathcal{W})$, because c is irreducible, and it is given by $p = (p_x)_{x \in S}$ with

$$p_x(\eta, v) = z_x^{-1}(\eta) \prod_{i=1}^{k-1} \frac{c_x(\eta, v_i, v_{i+1})}{c_x(\eta, v_{i+1}, v_i)}, \quad v \in W \setminus \{0\},$$
$$p_x(\eta, 0) = z_x^{-1}(\eta),$$

where $z_x^{-1}(\eta)$ is the normalizing factor, $k \geq 2$, and $0 = v_1, \ldots, v_k = v$ with $c_x(\eta, v_i, v_{i+1}) > 0$, $i = 1, \ldots, k-1$. Consequently,

$$\gamma_x^\nu(\pi_x^{-1}(v), \eta) = p_x(\eta, v), \quad \nu\text{-a.s.}, \; v \in W.$$

Since γ^ν is a specification, it is consistent. Hence p is ν-a.e. consistent, i.e. for any $x, y \in S$ and ν-a.a. $\eta \in \mathbb{X}$ with $\eta(x) = \eta(y) = 0$,

$$\begin{aligned} &p_x(\eta, v)\, p_y(\tau_x(\eta, v), u) p_x(\tau_y(\eta, u), 0) p_y(\eta, 0) \\ &\quad = p_y(\eta, u) p_x(\tau_y(\eta, u), v) p_y(\tau_x(\eta, v), 0) p_x(\eta, 0). \end{aligned} \tag{4.4}$$

By the preliminary considerations above, $p_x(\cdot, v)$ is a continuous function for any $v \in W$. Thus equation (4.4) extends to all $\eta \in \mathbb{X}$ because ν is dense. □

Example. Stochastic Ising models

We consider spin-flip processes, that are spin processes with $W = \{0, 1\}$. As before, $\mathcal{T}_0 = \{\{x\} : x \in S\}$. We abbreviate

$$c(x, \eta) := c_x(\eta, 1 - \eta(x)), \quad x \in S, \; \eta \in \mathbb{X},$$

for the rate of a transition $\eta \longrightarrow \eta^x$, where, for $z \in S$,

$$\eta^x(z) := \begin{cases} \eta(x), & z \neq x, \\ 1 - \eta(z), & z = x. \end{cases}$$

Suppose that the family $(c(x,\cdot))_{x\in S}$ is admissible and irreducible. Note that transition rates of spin-flip processes are irreducible if and only if they are positive.

Definition 4.2. The IPS corresponding to an admissible and positive family $(c(x,\cdot))_{x\in S}$ of rates is called *stochastic Ising model*, if there is a uniformly convergent potential $\Phi = (\Phi_A)_{A\in\mathcal{T}}$ such that for any $x \in S$, the expression

$$c(x,\eta)\exp\Big\{\sum_{A\ni x}\Phi_A(\eta)\Big\}, \quad \eta \in \mathbb{X}, \tag{DB-L}$$

does not depend on the coordinate $\eta(x)$.

Remark 4.1. Our notion of stochastic Ising model is slightly weaker than that of Liggett [14, Ch. IV]. More detailed, a spin-flip process is a *stochastic Ising model in the sense of Liggett* if there exists a collection $(J_R)_{R\in\mathcal{T}\cup\{\emptyset\}}$ of real numbers with

$$\sum_{R\ni x}|J_R| < \infty$$

such that

$$c(x,\eta)\exp\Big\{\sum_{R\ni x}J_R\chi_R(\eta)\Big\}, \quad \eta \in \mathbb{X}, \tag{DB-L*}$$

where

$$\chi_R(\eta) := \prod_{x\in R}[2\eta(x)-1], \quad R \in \mathcal{T},$$

$$\chi_\emptyset(\eta) := 1, \quad \eta \in \mathbb{X},$$

does not depend on the coordinate $\eta(x)$. Thus the notion of stochastic Ising model in Definition 4.2 is formulated with respect to a general uniformly convergent potential $\Phi = (\Phi_A)_{A\in\mathcal{T}}$, while that of Liggett concerns an absolutely summable so-called *spin potential.*

Obviously a stochastic Ising model in the sense of Liggett is one in the sense above. Conversely, if a spin-flip process is a stochastic Ising model with respect to some *finite range* potential Φ, then it is a stochastic Ising model in the sense of Liggett. Indeed, since $W = \{0,1\}$ is finite, the Hamiltonians

$$H_\Lambda^\Phi = \sum_{A\in\mathcal{T}, A\cap\Lambda\neq\emptyset}\Phi_A, \quad \Lambda \in \mathcal{T},$$

are bounded if the potential Φ is uniformly convergent. Therefore Theorem (2.35)(b) in [9] applies, which states that there is a uniformly convergent, so-called *lattice-gas potential* Φ^{gas} equivalent to Φ. If the latter is even absolutely summable, then, by [9, Ex. (2.38)], there exists an absolutely summable spin potential Φ^{spin} equivalent to Φ^{gas}. Under the assumption that Φ is of finite range, one finds that Φ^{gas} and Φ^{spin} are of finite range [19] and therefore absolutely summable. However, if the finite-range assumption is relaxed, then it is not clear, whether there is an *absolutely summable* spin potential which is equivalent to Φ. It was shown in [19] that even in the case that Φ is absolutely summable it is possible that there exists no equivalent absolutely summable spin potential.

Corollary 4.1. *A spin-flip process with admissible and positive rates c is a stochastic Ising model if and only if the condition (2.b) in Theorem 4.1 is satisfied.*

Any IPS which is a stochastic Ising model with respect to some potential Φ is globally reversible and satisfies

$$\mathscr{R}(c) = \mathscr{G}(\gamma^{\Phi}).$$

Proof. Suppose that we are given a spin-flip process with positive rates that satisfies condition (2.b) in Theorem 4.1. Then (DB) is solvable for all $\xi \in \mathbb{X}$ and, for fixed $x \in S$, $\xi \in \mathbb{X}$, the unique solution is given by

$$p_x(\xi, v) = \frac{c(x, \tau_x(\xi, 1-v))}{c(x, \tau_x(\xi, 1-v)) + c(x, \tau_x(\xi, v))}, \quad v \in W.$$

By Theorem 4.1, there is a positive and continuous specification $\gamma = (\gamma_T)_{T \in \mathcal{T}}$ such that

$$\gamma_x(\pi_x^{-1}(v), \xi) = p_x(\xi, v), \quad \xi \in \mathbb{X},\ v \in W,\ x \in S,$$

and

$$\mathscr{R}(c) = \mathscr{G}(\gamma) \neq \emptyset.$$

Further, by Proposition 2.5(2), there is a uniformly convergent vacuum potential Φ such that $\gamma = \gamma^{\Phi}$. Hence γ^{Φ} solves (DB). But γ^{Φ} satisfies

$$\gamma_x(\pi_x^{-1}(v), \xi) = \exp\Big\{ -\sum_{A \ni x} \Phi(\tau_x(\xi, v))\Big\} Z_x^{-1}(\xi), \quad \xi \in \mathbb{X},\ v \in W,\ x \in S,$$

where $Z_x^{-1}(\xi)$ is the normalizing factor. Inserting this representation into (DB) we get for $\xi \in \mathbb{X}$, $x \in S$,

$$c(x, \xi) \exp\Big\{ -\sum_{A \ni x} \Phi(\xi)\Big\} Z_x^{-1}(\xi) = c(x, \xi^x) \exp\Big\{ -\sum_{A \ni x} \Phi(\xi^x)\Big\} Z_x^{-1}(\xi^x).$$

Since $Z_x(\xi^x) = Z_x(\xi)$, the normalizing factor can be canceled on both sides of the equation. But this implies (DB-L). □

Remark 4.2.

(1) A sufficient condition for an IPS to be a stochastic Ising model was given in [14, Thm. IV.2.13]: If the IPS assigned to a positive, *finite-range* family $(c(x,\cdot))$ is globally reversible, then it is a stochastic Ising model, i.e.

$$\mathscr{R}(c) \neq \emptyset, \ c \text{ positive} \quad \Longrightarrow \quad \text{(DB-L}^*\text{)}$$

for some finite range potential Φ.

(2) The analogous statement to our Corollary 3.1 for spin-flip processes is [14, Prop. IV.2.7]:

$$\nu \in \mathscr{R}$$
$$\Updownarrow$$
$$\rho_x(\eta) := \nu\big(\pi_x^{-1}(\eta(x)) \mid C_x\big)(\eta) = \frac{c(x,\eta^x)}{c(x,\eta) + c(x,\eta^x)}, \qquad \eta \in \mathbb{X}, \ x \in S.$$

(3) Our findings are conform with the results of Mu Fa Chen and coworkers which were reported in [1]. There the global reversibility of spin processes (and exchange processes) with $W = \{0,1\}$ is studied with the help of so-called potentiality.

Example. Alignment model

The alignment model was introduced in [13] as a model for the emergence of a global orientation from local alignment for cell populations. It is a spin process with state space $W = \{\pm e_i, i = 1,\dots,d\}$, where e_i, $i = 1,\dots,d$ are the unit vectors of $\mathbb{R}^d$. The transition rates are given by

$$c_x(\eta,u) := \exp\Big\{\gamma \sum_{z:|z-x|=1} \eta(z) \circ u\Big\}, \qquad x \in S, \ \eta \in \mathbb{X}, \ u \in W,$$

where $\gamma > 0$ is a parameter called sensitivity and $\circ$ denotes the scalar product in $\mathbb{R}^d$. We observe that the family $c = (c_x(\cdot,\cdot))_{x\in S}$ is positive. For $\eta \in \mathbb{X}$, $x \in S$, the rates $c_x(\eta,u,v) := c(\tau_x(\eta,u),v)$, $u,v \in W$, are irreducible. Since $c_x(\cdot,v)$ is C_x-measurable for each $x \in S$, $v \in W$, (DB) is solvable for all $\xi \in \mathbb{X}$.

By Proposition 4.2, the solution is given by

$$\begin{aligned}
p_x(\eta,v) &= \frac{1}{1 + \sum_{u\in W\setminus\{v\}} c_x(\eta,v,u)/c_x(\eta,u,v)} \\
&= \frac{1}{1 + \sum_{u\in W\setminus\{v\}} \exp\big\{\sum_{z:|z-x|=1} \eta(z)\circ u - \sum_{z:|z-x|=1} \eta(z)\circ v\big\}} \\
&= Z_x^{-1}(\eta) \exp\Big\{\sum_{z:|z-x|=1} \eta(z) \circ v\Big\}
\end{aligned}$$

where

$$Z_x(\eta) = \sum_{u \in W} \exp\Big\{ \sum_{z:|z-x|=1} \eta(z) \circ u \Big\}$$

and $x \in S$, $v \in W$, $\eta \in \mathbb{X}$. One easily finds that

$$p_x(\eta, v) = \exp\Big\{ - \sum_{A \ni x} \Phi(\tau_x(\eta, v)) \Big\},$$

where $\Phi = (\Phi_T)_{T \in \mathcal{T}}$ with

$$\Phi_T(\eta) := \begin{cases} \eta(x) \circ \eta(y), & \text{if } T = \{x, y\}, \text{ with } |x - y| = 1, \\ 0, & \text{otherwise}, \end{cases}$$

is of finite range. Thus the specification γ^Φ solves (DB). Applying Corollary 3.1, we find that $\mathscr{R}(c) = \mathscr{G}(\gamma^\Phi)$. The subfamily $(\gamma^\Phi)^0 = (\gamma_x^\Phi)_{x \in S}$ is a positive and continuous one-point specification, hence $\mathscr{G}(\gamma^\Phi) = \mathscr{G}((\gamma^\Phi)^0)$.

4.2. Composite IPS

We consider IPS that are composed of several types of local dynamics like spin-flip and spin-exchange processes. Let $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}$ with $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$ and suppose that $c_{\mathcal{T}_1} = (c_T(\cdot, \cdot))_{T \in \mathcal{T}_1}$ and $c_{\mathcal{T}_2} = (c_T(\cdot, \cdot))_{T \in \mathcal{T}_2}$ are two admissible families of transition rate functions. Define $\mathcal{T}_0 := \mathcal{T}_1 \cup \mathcal{T}_2$ and $c_{\mathcal{T}_0} := (c_T(\cdot, \cdot))_{T \in \mathcal{T}_0}$. Each $c_{\mathcal{T}_i}$, $i = 0, 1, 2$, can be understood as a family of transition rate functions where the local dynamics on $\mathcal{T} \setminus \mathcal{T}_i$ is trivial, i.e. the corresponding rates vanish. The following proposition is a direct corollary of Theorem 3.1.

Proposition 4.3. *Let $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2$ where $\mathcal{T}_1, \mathcal{T}_2 \subset \mathcal{T}$ with $\mathcal{T}_1 \cap \mathcal{T}_2 = \emptyset$. Suppose that $c_{\mathcal{T}_1} = (c_T(\cdot, \cdot))_{T \in \mathcal{T}_1}$ and $c_{\mathcal{T}_2} = (c_T(\cdot, \cdot))_{T \in \mathcal{T}_2}$ are finite and standard. Then*

$$\mathscr{R}(c_{\mathcal{T}_0}) = \mathscr{R}(c_{\mathcal{T}_1}) \cap \mathscr{R}(c_{\mathcal{T}_2}).$$

Proof. According to Theorem 3.1 it holds for $i = 0, 1, 2$ that $\nu \in \mathscr{R}(c_{\mathcal{T}_i})$ if and only if

$$\nu(gA_T f - fA_T g) = 0, \quad T \in \mathcal{T}_i,\ f, g \in C(\mathbb{X}).$$

Hence $\nu \in \mathscr{R}(c_{\mathcal{T}_0})$ if and only if $\nu \in \mathscr{R}(c_{\mathcal{T}_i})$ for $i = 1, 2$. □

Next we specialize to IPS that are composed of spin-flips and more complex local mechanisms, i.e. we set $\mathcal{T}_1 := \{\{x\} : x \in S\}$.

Proposition 4.4. *Suppose that $c = (c(\cdot, \cdot))_{T \in \mathcal{T}}$ is a family of admissible, finite and standard transition rates with $\mathcal{T}_1 = \{\{x\} : x \in S\} \subset \mathcal{T}_0$. Assume that the family $(c_x(\cdot, \cdot))_{x \in S}$ is irreducible and satisfies condition (2) of Theorem 4.1. Let the positive specification which is determined from $(c_x(\cdot, \cdot))_{x \in S}$ according to Theorem 4.1 be denoted by $\gamma = (\gamma_T)_{T \in \mathcal{T}}$. Then the following statements are equivalent:*

(1) $\mathscr{R}(c) = \mathscr{G}(\gamma) = \{\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F}) : \nu\gamma_x = \nu\} \neq \emptyset$.

(2) *For each* $T \in \mathcal{T}_0 \setminus \mathcal{T}_1$ *and each* $u, v \in \mathbb{X}_T$, $\eta \in \mathbb{X}$, *it holds that*

$$\gamma_T(\eta, \pi_T^{-1}(u))c_T(\eta, u, v) = \gamma_T(\eta, \pi_T^{-1}(v))c_T(\eta, v, u).$$

Proof. Denote $\mathcal{T}_2 := \mathcal{T} \setminus \mathcal{T}_1$ and $c_{\mathcal{T}_i} := (c_T(\cdot, \cdot))_{T \in \mathcal{T}_i}$, $i = 1, 2$. By Proposition 4.3 $\mathscr{R}(c) = \mathscr{R}(c_{\mathcal{T}_1}) \cap \mathscr{R}(c_{\mathcal{T}_2})$ and by Theorem 3.1

$$\mathscr{R}(c_{\mathcal{T}_1}) = \mathscr{G}(\gamma) = \{\nu \in \mathscr{P}(\mathbb{X}, \mathcal{F}) : \nu\gamma_x = \nu\} \neq \emptyset,$$

where $\gamma = (\gamma_T)_{T \in \mathcal{T}}$ is the specification which is determined from $(c_x(\cdot, \cdot))_{x \in S}$ according to Theorem 4.1. Hence $\mathscr{R}(c) = \mathscr{G}(\gamma) \cap \mathscr{R}(c_{\mathcal{T}_2})$, and it remains to show that the condition $\mathscr{G}(\gamma) \subset \mathscr{R}(c_{\mathcal{T}_2})$ is equivalent to (2).

Suppose that $\mathscr{G}(\gamma) \subset \mathscr{R}(c_{\mathcal{T}_2})$ and fix $\nu \in \mathscr{G}(\gamma)$. Then it follows from Corollary 3.1(1) applied to $c_{\mathcal{T}_2}$ that ν-a.s.

$$\nu(\pi_T^{-1}(u) \mid \mathcal{C}_T)c_T(\cdot, u, v) = \nu(\pi_T^{-1}(v) \mid \mathcal{C}_T)c_T(\cdot, v, u), \quad u, v \in \mathbb{X}_T,\ T \in \mathcal{T}_2,$$

and therefore ν-a.s.

$$\gamma_T(\cdot, \pi_T^{-1}(u))c_T(\cdot, u, v) = \gamma_T(\cdot, \pi_T^{-1}(v))c_T(\cdot, v, u), \quad u, v \in \mathbb{X}_T,\ T \in \mathcal{T}.$$

Since γ is positive and continuous, the measure ν is dense (Proposition 2.2). The functions $c_T(\cdot, u, v)$, $u, v \in \mathbb{X}_T$, $T \in \mathcal{T}_2$ are continuous, as well, consequently the latter equation holds for all $\eta \in \mathbb{X}$ which is (2).

Conversely, suppose that (2) is satisfied. Then we get from Corollary 3.1(2) applied to $c_{\mathcal{T}_2}$ that $\mathscr{G}(\gamma) \subset \mathscr{R}(c_{\mathcal{T}_2})$. □

Example. Processes with combined Glauber- and Kawasaki-type dynamics

Let $W = \{0, 1\}$ and suppose that we are given a family c of admissible rate functions. Let further a finite-range potential $\Phi = (\Phi_A)_{A \in \mathcal{T}}$ be given with corresponding *Hamiltonians*

$$H_\Lambda^\Phi(\eta) := \sum_{A \in \mathcal{T}, A \cap \Lambda \neq \emptyset} \Phi_A(\eta), \quad \eta \in \mathbb{X},\ \Lambda \in \mathcal{T}.$$

Assume that $c_{\{x,y\}}(\eta, u) = 0$, $x, y \in S$, $\eta \in \mathbb{X}$, $u \in \mathbb{X}_{\{x,y\}}$ unless $|x - y| = 1$ and $\eta(x) = u(y)$, $\eta(y) = u(x) = 0$. Then, using the notations

$$\eta^x(z) = \begin{cases} 1 - \eta(x), & z = x, \\ \eta(z), & \text{otherwise,} \end{cases}$$

and

$$\eta^{xy}(z) = \begin{cases} \eta(x), & z = y, \\ \eta(y), & z = x, \\ \eta(z), & \text{otherwise}, \end{cases}$$

the rates $c_T(\cdot,\cdot)$ are for $T = \{x\} \in \mathcal{T}_1$ completely described by $c(x,\eta) := c_x(\eta, 1-\eta(x))$ and for $T = \{x,y\} \in \mathcal{T}_2$ by $c(x,y,\eta) := c_{\{x,y\}}(\eta, \eta^{xy}_{\{x,y\}})$.

If $c(x,\eta) > 0$ for each $x \in S$, $\eta \in \mathbb{X}$ and the equation

$$c(x,\eta)\exp\{-H_x(\eta)\} = c(x,\eta^x)\exp\{-H_x(\eta^x)\} \tag{4.5}$$

is satisfied for each $x \in S$, $\eta \in \mathbb{X}$, then there is a globally reversible measure if and only if the rates $c(x,y,\eta)$ satisfy for $x,y \in S$, $\eta \in \mathbb{X}$ the condition

$$c(x,y,\eta)\exp\{-H_{\{x,y\}}(\eta)\} = c(x,y,\eta^{xy})\exp\{-H_{\{x,y\}}(\eta^{xy})\}. \tag{4.6}$$

If the latter condition holds, then $\mathscr{R}(c) = \mathscr{G}(\gamma^\Phi)$.

Indeed, since

$$\gamma^\Phi_T(\eta, \pi_T^{-1}(u)) = \left(Z^\Phi_T(\eta)\right)^{-1}\exp\{-H^\Phi_T(\tau_T(\eta,u))\}, \quad T \in T,\ \eta \in \mathbb{X},\ u \in \mathbb{X}_T,$$

where the partition function $Z^\Phi_T(\cdot)$ does not depend on the coordinates in T, the equations (4.5) and (4.6) are equivalent to the statement that γ^Φ solves (DB) for $(c_x(\cdot,\cdot))_{x\in S}$ and $(c_T(\cdot,\cdot))_{T\in\mathcal{T}_2}$, respectively. Since γ^Φ is a specification, it is consistent, hence condition (2) of Theorem 4.1 is satisfied. Applying Proposition 4.4, we get the above statement.

Example. Collective migration model

The collective migration model (CMM) was introduced in [13] as a model for the emergence of global swarms from local alignment of the preferred directions of cell movement within cell populations. In the CMM, we have

$$\mathcal{T}_0 = \big\{\{x\} : x \in S\} \cup \{\{x,y\} : x,y \in S, |x-y| = 1\big\},$$
$$W = \{0, \uparrow, \downarrow, \rightarrow, \leftarrow\}.$$

Here the interpretation is that $\eta(x) = 0$ if lattice site $x \in S$ is not occupied. If $\eta(x) \neq 0$, then the lattice site x is occupied by a cell which has the preferred direction of movement $\eta(x)$. Transitions occur with rates

$$c_x(\eta,u) := \exp\left\{\gamma \sum_{z:|z-x|=1} \eta(z)\circ u\right\}, \quad x \in S,\ \eta \in \mathbb{X},\ u \in W,$$

where $\gamma > 0$ is a parameter called sensitivity and $\circ$ denotes the scalar product in $\mathbb{R}^d$, resp.

$$c_{\{x,y\}}(\eta,u) = \begin{cases} m, & \eta(x) = u(y) = y-x,\ \eta(y) = u(x) = 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $x, y \in S$, $|x - y| = 1$, $\eta \in \mathbb{X}$, $u \in \mathbb{X}_{\{x,y\}}$ and m is a non-negative parameter, the migration parameter.

It is easily checked that, for any $T = \{x, y\} \in \mathcal{T}_0$ and arbitrary $\eta \in \mathbb{X}$, (DB) cannot be satisfied. Thus there is no globally reversible measure for the collective migration model.

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