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$$\lim_{n\to\infty} \underbrace{v(\ell_n)\,\ell_n^2} = \lim_{n\to\infty}\left(\ell_n \sum_{k=1}^{n} \min_{x\in I_k(n)} f(x)\right)$$

$$= \underbrace{\left(\sum_{k=1}^{n}\left\lfloor \frac{1}{\ell_n}\min_{x\in I_k(n)} f(x)\right\rfloor\right)\ell_n^2}_{=:c_n}$$

- $$c_n \le \left(\sum_{k=1}^{n} \frac{1}{\ell_n}\min_{x\in I_k(n)} f(x)\right)\ell_n^2 = \underbrace{\frac{1}{\ell_n}\cdot \ell_n^2}_{=\ell_n}\cdot \sum_{k=1}^{n}\min_{x\in I_k(n)} f(x)$$

- $$c_n \ge \left(\sum_{k=1}^{n}\left(\frac{1}{\ell_n}\min_{x\in I_k(n)} f(x) - 1\right)\right)\ell_n^2$$

$$= \left(\sum_{k=1}^{n} \frac{1}{\ell_n}\min_{x\in I_k(n)} f(x) - \underbrace{\sum_{k=1}^{n} 1}_{=n}\right)\ell_n^2$$

$$= \underbrace{\left(\sum_{k=1}^{n} \frac{1}{\ell_n}\min_{x\in I_k(n)} f(x)\right)\ell_n^2}_{=\ell_n \sum_{k=1}^{n}\min_{x\in I_k(n)} f(x)} - \underbrace{n\,\ell_n^2}_{=n\left[\frac{b}{n}\right]^2 \le n\frac{b^2}{n^2} = \frac{b^2}{n}}$$

$$\Rightarrow \lim_{n\to\infty} c_n \ge \lim_{n\to\infty}\left(\ell_n \sum_{k=1}^{n}\min_{x\in I_k(n)} f(x)\right) - \underbrace{\lim_{n\to\infty}\frac{b^2}{n}}_{=0}$$