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$$T_n := \sum_{k=1}^n a_k X_k, \quad \sum_{k=1}^n a_k = 1, \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k^2 = 0$$

zu zeigen: $\mathbb{E} T_n = \mu$

$$\begin{aligned} \mathbb{E}(\sum a_k X_k) &= \sum \mathbb{E}(a_k X_k) \\ &= \sum a_k \underbrace{\mathbb{E} X_k}_{=\mu} = \sum a_k \mu \\ &= \mu \sum a_k = \mu \end{aligned}$$

zu zeigen: $\text{Var } T_n \xrightarrow{n \rightarrow \infty} 0$

$$\text{Var } T_n = \text{Var}(\sum a_k X_k)$$

$$X_k \text{ unabh. } \Downarrow \sum \text{Var}(a_k X_k) = \sum a_k^2 \underbrace{\text{Var } X_k}_{=: c \text{ (unabh. von } k)}}$$

$$= c \underbrace{\sum a_k^2}_{\xrightarrow{n \rightarrow \infty} 0} \xrightarrow{n \rightarrow \infty} 0$$

Bsp:

$$a_k := \frac{1}{n}, \quad \sum_{k=1}^n \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

$$\sum a_k^2 = \sum \frac{1}{n^2} = n \cdot \frac{1}{n^2} = \frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$$

Bsp.:

$$a_1 = \frac{1}{2n}, \quad a_k = \frac{1}{n} + \frac{1}{2n(n-1)} \quad (k \geq 2)$$

$$\begin{aligned} \sum a_k &= \frac{1}{2n} + (n-1) \left(\frac{1}{n} + \frac{1}{2n(n-1)} \right) \\ &= \frac{2(n-1) + 1}{2n(n-1)} \end{aligned}$$

$$= \frac{1}{2n} + \frac{2n-2+1}{2n} = \frac{2n}{2n} = 1$$

$$\begin{aligned} \sum a_k^2 &= \left(\frac{1}{2n} \right)^2 + (n-1) \left(\frac{1}{n} + \frac{1}{2n(n-1)} \right)^2 \\ &= \left(\frac{2n-1}{2n(n-1)} \right)^2 \end{aligned}$$

$$\begin{aligned} &= \underbrace{\left(\frac{1}{2n} \right)^2}_{\rightarrow 0} + \frac{(2n-1)^2}{(2n)^2(n-1)} \\ &= \frac{n^2 \left(2 - \frac{1}{n} \right)^2}{n^3 \cdot 4 \left(1 - \frac{1}{n} \right)} = \underbrace{\frac{1}{n}}_{\rightarrow 0} \cdot \underbrace{\frac{\left(2 - \frac{1}{n} \right)^2}{4 \left(1 - \frac{1}{n} \right)}}_{\rightarrow 1} \end{aligned}$$

$$\xrightarrow{n \rightarrow \infty} 0$$